Separation axioms via \(\delta\)-set in topological vector spaces

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### Abstract
In this paper we introduce a new sort of spaces as \(\delta\)-Homogenous space, \(\delta\)-Hausdorff space and \(\delta\)-Compact space. It provides a new connection between \(\delta\)-Vector spaces and \(\delta\)-homogenous spaces. Also we investigated the relationship between the translation and scalar multiplication mappings and \(\delta\)-homeomorphism on \(\delta\)-Topological vector spaces. Finally we derive \(\delta\)-Topological vector space is \(\delta\)-Hausdorff and \(\delta\)-Compact spaces.

### Keywords
\(\delta\)-topological vector spaces, \(\delta\)-homeomorphism, \(\delta\)-continuous, \(\delta\)-Hausdorff, \(\delta\)-compact.

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54A05, 54C05, 54C08, 54N17.

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### 1. Introduction
A topological space is a vector space with a topological structure such that the algebraic operations addition and scalar multiplication are continuous. The concept of vector spaces was introduced by Kolmogroff \([1]\). In 2015, Khan et al \([2]\) introduced the \(s\)-topological vector spaces which are generalization of topological vector spaces. In 2016 Khan and Inqbal \([3]\) introduced the irresolute independent of topological vector spaces. In 2019, \(\beta\)-topological vector spaces have been introduced by Sharma and M.Ram \([8]\). In 2019, S.Sharma et al. \([9]\) investigated almost \(\beta\)-topological vector spaces. Maki et al \([4]\) introduced the notions of generalized homeomorphism in topological spaces. In this paper we introduce a new sort of spaces as \(\delta\)-Homogenous space, \(\delta\)-Hausdorff space and \(\delta\)-Compact space. It provides a new connection between \(\delta\)-Vector spaces and \(\delta\)-homogenous spaces. Also we investigated the relationship between the translation and scalar multiplication mappings and \(\delta\)-homeomorphism on \(\delta\)-Topological vector spaces. Finally we derive \(\delta\)-Topological vector space is \(\delta\)-Hausdorff and \(\delta\)-Compact spaces.

### 2. Preliminaries
In this section, we recall some definitions and basic results of fractional calculus which will be used throughout the paper.

**Definition 2.1.** \([7]\) A subset \(A\) of a topological space \((X, \tau)\) is

(i) Regular\(\delta\)-open if \(A = \text{int}(\text{cl}^\delta(A))\)

(ii) Regular\(\delta\)-closed if \(A = \text{cl}(\text{int}^\delta(A))\)

**Definition 2.2.** \([5]\) The \(\delta\)-interior of a subset \(A\) of \(X\) is called \(\delta\)-open if \(A = \text{int}_\delta(A)\) i.e., a set is if it is the union of Regular\(\delta\)-open sets. The complement of a \(\delta\)-open set is called \(\delta\)-closed set in \(X\).
Definition 2.3. [6] A $\delta$-topological vector space is a vector space $X$ over the field $F$ (real or complex) with a topology $\tau$ with the following conditions.

(i) Vector addition mapping $m : X \to Y$ defined by $m((x,y)) = x+y$, for each $x,y$ in $X$ is $\delta$-continuous

(ii) Scalar multiplication mapping $M : F \times X \to X$ which define by $M((\lambda, x)) = \lambda x$ for each $\lambda$ in $F$ and $x,y$ in $X$ is $\delta$-continuous.

The pair $(X,F)$ is said to be Topological vector space. In short, it is denoted by $X$, a $\delta$-TVS.

Definition 2.4. [2] If $X$ is a Vector space then $\delta$ denotes its identity element, and for a fixed $x \in X$, $X : X \to X; x \to x+y$ and $T_x : X \to X; y \to y+x$ denote the left and right translation by $x$ respectively.

Definition 2.5. [6] Let $Y$ be a Linear Subspace of $(X, \tau)$ which means $Y + Y \subseteq Y$ and for all $\alpha \in F, \alpha Y \subseteq Y$.

Result 2.6. [6] Let $(X,F)$ be a $\delta$-TVS. If $A$ is open in $(X,F)$, then the following are true.

(i) $x + A$ is a $\delta$-open for each $x \in X$

(ii) $\alpha A$ is a $\delta$-open for all non-zero scalar $\alpha$ in $X$.

Result 2.7. [6] In a $\delta$-TVS $(X,F)$, for any $\delta$-open set $U$ containing $0$, there exists a symmetric $\delta$-open set $V$ containing $0$ such that $V + V \subseteq U$.

Result 2.8. [6] Let $X$ be $\delta$-TVS. If $A$ is open subset of $X$ then $A + B$ is a $\delta$-open in $X$ for any subset $B$ of $X$.

3. Translation Mappings

In this section we prove that translation mappings are $\delta$-continuous in a $\delta$-Topological Vector Spaces. Also it’s basic properties have been derived.

Theorem 3.1. In a $\delta$-TVS $(X,F)$, for any $x \in X$, the translation mapping $T_x : X \to X$ defined by $T_x(y) = y + x$ for all $y \in X$ is $\delta$-continuous function.

Proof. Suppose that $(X,F)$ is a $\delta$-topological vector space. Let $x \in X$ be arbitrary. Let $K$ be any open set in the codomain $X$ containing $T_x(y) = y + x$. By hypothesis, there exists a $\delta$-open set $U$ containing $y$ and $V$ containing $x$ such that $U + V \subseteq K$. Then $T_x(U) = U + x \subseteq U + V \subseteq K$. It is proved that for every open set $K$ containing $T_x(y)$, $\exists$ a $\delta$-open set $U$ containing $y$ such that $T_x(U) \subseteq K$. Therefore the translation mapping $T_x$ is $\delta$-continuous function.

Theorem 3.2. In a $\delta$-TVS $(X,F)$, for any $\alpha \in F$, the multiplication mapping $M_\alpha : X \to X$ defined by $M_\alpha(x) = \alpha x$ is $\delta$-continuous mapping.

Proof. Suppose that $(X,F)$ is a $\delta$-topological vector space. Let $K$ be any open set in the $X$ containing $M_\alpha(x) = \alpha x$. By hypothesis, there exists a $\delta$-open set $U$ in $F$ containing $\alpha$ and $V$ in $X$ containing $x$ such that $UV \subseteq K$. Then $M_\alpha(UV) = \alpha V \subseteq UV \subseteq K$. It is proved that for every open set $K$ containing $M_\alpha(x)$, $\exists$ a $\delta$-open set $V$ in $X$ containing $x$ such that $M_\alpha(V) \subseteq K$. Hence $M_\alpha$ is $\delta$-continuous mapping.

Theorem 3.3. Let $(X,F)$ be a $\delta$-TVS. If $U$ is open in $X$, then $U + x$ is a $\delta$-open subset of $X, \forall x \in X$.

Proof. Let $u + x \in U + x$ be arbitrary. Now $U$ is open set in $X$ containing $u = u + x + x = T_x(U + x)$. Since the translation map $T_x$ is $\delta$-continuous, $\exists$ a $\delta$-open set $V$ containing $u + x$ such that $T_x(V) \subseteq U + x$. That is, $V + (-x) \subseteq U$ and hence $V \subseteq U + x$. It is proved that for any point $u + x \in U + x$, there exists $\delta$-open set $V$ containing $u + x$ such that $u + x \in V \subseteq U + x$. Therefore $u + x$ is $\delta$-open subset of $X, \forall x \in X$.

Theorem 3.4. Let $(X,F)$ be a $\delta$-TVS. If $U$ is open in $X$, then $\alpha U$ is a $\delta$-open in $X$ for any nonzero element $\alpha \in F$.

Proof. Let $x \in \alpha U$ be arbitrary. Then $x = \alpha u$ for some $u \in U$. Now $U$ is open set in the codomain $X$ containing $u = \frac{1}{\alpha}(\alpha u) = M_1(\alpha U) = M_1(x)$. Since the multiplication mapping $M_1 : X \to X$ is $\delta$-continuous, there exists a $\delta$-open set $V$ containing $\alpha u = x$ such that $M_1(V) \subseteq U$. That is, $\frac{1}{\alpha}(V) \subseteq U$. Hence $V \subseteq \alpha U$. Therefore $\alpha u$ is $\delta$-open subset of $X$ for any non-zero element $\alpha \in F$.

4. $\delta$-Closure in a $\delta$-TVS

Definition 4.1. The $\delta$-interior of a subset $A$ of $X$ is the union of all regular $\delta$-open sets of $X$ contained in $A$ and is denoted by $int_\delta(A)$.

Definition 4.2. A subset $A$ of a topological space $(X, \tau)$ is called $\delta$-open if $A = int_\delta(A)$, i.e., a set is $\delta$-open if it is the union of regular $\delta$-open sets. The complement of a $\delta$-open is called $\delta$-closed set in $X$.

Note 4.3. The $\delta$-closure of a subset $A$ of $(X, \tau)$ is denoted by $cl_\delta(A)$.

Theorem 4.4. In a $\delta$-TVS $(X,F)$, a scalar multiple of a $\delta$-closed set is $\delta$-closed for any $\alpha \in F$.

Proof. Let $U$ be any $\delta$-closed subset of $X$ and $\alpha \in F$ be arbitrary. $(\alpha U)^c = X \setminus \alpha U = \alpha(X \setminus U) = \alpha U^c$. Since $U$ is $\delta$-closed subset of $X$, $U^c$ is $\delta$-open subset of $X$. Since every $\delta$-open set is open, $U^c$ is an open subset of $X$. By Result 2.6, $\alpha U^c$ is a $\delta$-open subset of $X$. Then $(\alpha U)^c$ is a $\delta$-open. So $\alpha U$ is a $\delta$-closed subset of $X$.

Theorem 4.5. Let $A$ be any closed subset of a $\delta$-topological vector space $(X,F)$. Then the following are true.

(i) $x + A$ is $\delta$-closed for each $x \in X$

(ii) $\alpha A$ is a $\delta$-closed for each non-zero scalar $\alpha$ in $F$. 


Proof. (i) Let \( y \in cl_{\delta}(x + A) \). Now consider \( z = -x + y \) and let \( K \) be any open set in \( X \) containing \( z \). Then by definition of \(*\delta\)-topological vector space, there exists \(*\delta\)-open sets \( U \) and \( V \) in \( X \) such that \(-x \in U, y \in V \) and \( U + V \subseteq K \). Since \( y \in cl_{\delta}(x + A) \), \( (x + A) \cap V \neq \emptyset \). Then there is \( a \in (x + A) \cap V \). Now \(-x + a \in A \cap (U + V) \subseteq A \cap K \Rightarrow A \cap K \neq \emptyset \) which implies \( z \in cl_{\delta}(A) = y \in x + A \). Hence \( cl_{\delta}(x + A) \subseteq x + A \). Always \( x + A \subseteq cl_{\delta}(x + A) \). Thus \( x + A = cl_{\delta}(x + A) \). Hence \( x + A \) is \(*\delta\)-closed in \( X \).

(ii) Let \( x \in cl_{\delta}(A) \) and let \( K \) be any open neighborhood of \( y = \frac{1}{\alpha}x \) in \( X \). Since \((X,F,\tau)\) is \(*\delta\)-TVS, \( \exists \delta\)-open sets \( U \in F \) containing \( \frac{1}{\alpha}x \) and \( V \) in \( X \) containing \( x \) such that \( U, V \subseteq K \). By hypothesis, \( \langle \alpha A \rangle \cap V \neq \emptyset \). Therefore there is \( a \in \langle \alpha A \rangle \cap V \). Now \( \frac{1}{\alpha}a \in A \cap \langle \alpha V \rangle \subseteq A \cap K \Rightarrow A \cap K \neq \emptyset \) which implies \( y \in cl_{\delta}(A) = A \Rightarrow x \in \alpha A \). Hence \( cl_{\delta}(\alpha A) \subseteq \alpha A \). Always \( \alpha A \subseteq cl_{\delta}(\alpha A) \). Hence \( \alpha A = cl_{\delta}(\alpha A) \). Thus \( \alpha A \) is \(*\delta\)-closed in \( X \).

Theorem 4.6. Let \((X,F,\tau)\) be a \(*\delta\)-TVS. If \( U \) is \(*\delta\)-open set in \( X \), then there exists a \(*\delta\)-open set \( V \) in \( X \) containing \( 0 \) such that \( u + V \subseteq U \) for all \( u \in U \).

Proof. Let \( U \) be any \(*\delta\)-open set in \((X,F,\tau)\). Since every \(*\delta\)-open set is open, \( U \) is an open subset of \( X \). By Result 2.6, \( U + x \) is \(*\delta\)-open set in \( X \) for all \( x \in X \). In particular \( U - u \) is a \(*\delta\)-open set in \( X \) containing \( 0 \) for all \( u \in U \). By taking \( V = U - u \), we get a \(*\delta\)-open set \( V \) containing \( 0 \) such that \( u + V \subseteq U \).

Theorem 4.7. Let \( S \) and \( T \) be any subsets of a \(*\delta\)-TVS \((X,F,\tau)\), then \( cl_{\delta}(S) + cl_{\delta}(T) \subseteq cl_{\delta}(S + T) \).

Proof. Let \( z \in cl_{\delta}(S) + cl_{\delta}(T) \). Let \( S \) be an \(*\delta\)-open set in \( X \) containing \( z \). Since \( Y \) is \(*\delta\)-TVS, the condition of \(*\delta\)-Topological vector space, there exists \(*\delta\)-open sets \( U \) in \( X \) containing \( z \) in \( X \) such that \( U + V \subseteq K \). Since \( y \in cl_{\delta}(S) \), \( y \in cl_{\delta}(T) \), there are \( a \in S \cap U \) and \( b \in T \cap V \). Then \( a + b \in (S + T) \cap (U + V) \subseteq (S + T) \cap K \). So, \( K \cap (S + T) \neq \emptyset \). Therefore \( x + y \in cl_{\delta}(S + T) \).

Theorem 4.8. Let \((X,F,\tau)\) be a \(*\delta\)-TVS and let \( S, T \) be subsets of \((X,F,\tau)\). If \( T \) is \(*\delta\)-open, then \( S + T = cl_{\delta}(S + T) \).

Proof. Let \( S \) and \( T \) be any two subsets of a \(*\delta\)-TVS \( X \). Always \( S \subseteq cl_{\delta}(S) \). So \( S + T \subseteq cl_{\delta}(S + T) \). Now let \( y \in cl_{\delta}(S) + cl_{\delta}(T) \). Let \( S \) be \(*\delta\)-open set in \( X \) containing \( z \). Since \( Y \) is \(*\delta\)-TVS, the condition of \(*\delta\)-Topological vector space, there exists \(*\delta\)-open sets \( U \) in \( X \) containing \( z \) in \( X \) such that \( U + V \subseteq K \). Since \( y \in cl_{\delta}(S) \), \( y \in cl_{\delta}(T) \), there are \( a \in S \cap U \) and \( b \in T \cap V \). Then \( a + b \in (S + T) \cap (U + V) \subseteq (S + T) \cap K \). So, \( K \cap (S + T) \neq \emptyset \). Therefore \( x + y \in cl_{\delta}(S + T) \).

Theorem 5.1. A bijective function \( f \) from a \(*\delta\)-TVS \( X \) to itself is called \(*\delta\)-homeomorphism if \( f \) and \( f^{-1} \) are \(*\delta\)-continuous on a \(*\delta\)-TVS.

Theorem 5.2. A TVS \((X,F,\tau)\) is called \(*\delta\)-homogeneous space, if for all \( x, y \in X \), there is \(*\delta\)-homeomorphism \( f \) of the space \( X \) onto itself such that \( f(x) = y \).

Theorem 5.3. Translation mapping on a \(*\delta\)-topological vector space is \(*\delta\)-homeomorphism.

Proof. Let \((X,F,\tau)\) be a \(*\delta\)-TVS, \( \forall x \in X \), translation mapping \( T_x : X \rightarrow X \) is defined by \( T_x(z) = z + x \) for all \( z \in X \). Clearly, \( T_x \) is a bijective mapping for all \( x \in X \). By Theorem 3.1, \( T_x \) is \(*\delta\)-continuous. Let \( U \) be any open set containing the point \( z \), where \( z \in X \). By Theorem 3.3, \( U + x = T_x(U) \) is \(*\delta\)-open in \( X \). Therefore \( T_x \) is a \(*\delta\)-homeomorphism.

Theorem 5.4. Multiplication mapping on a \(*\delta\)-TVS is \(*\delta\)-homeomorphism.

Proof. Let \((X,F,\tau)\) be a \(*\delta\)-TVS and let the arbitrary scalar \( \alpha \in F \). Multiplication mapping \( M_\alpha : X \rightarrow X \) is \( M_\alpha(x) = \alpha x \). Obviously, it is a bijective mapping. By Theorem 3.2, \( M_\alpha \) is \(*\delta\)-continuous for any \( \alpha \in F \). Then \( M_\alpha(U) = \alpha U \) where \( U \) is any open set in \( X \). By Theorem 3.4, \( \alpha U \) is \(*\delta\)-open in \( X \). Hence \( M_\alpha \) is \(*\delta\)-homeomorphism.

Theorem 5.5. \(*\delta\)-closure of a linear subspace of a \(*\delta\)-TVS is a \(*\delta\)-TVS.

Proof. Let \((X,F,\tau)\) be a \(*\delta\)-TVS and \( H \) be any linear subspace of \( X \). Then \( H + H \subseteq \alpha H \) and \( \alpha H \subseteq H \) for all \( \alpha \in F \). So \( cl_{\delta}(H + H) \subseteq cl_{\delta}(H) \) and \( cl_{\delta}(\alpha H) \subseteq cl_{\delta}(H) \) for all \( \alpha \in F \). By Theorem 4.7, \( cl_{\delta}(H) + cl_{\delta}(H) \subseteq cl_{\delta}(H + H) \subseteq cl_{\delta}(H) \). Also since scalar multiplication is a \(*\delta\)-homeomorphism, by Theorem 4.4, it maps \(*\delta\)-closure of a set into \(*\delta\)-closure of its image. So \( cl_{\delta}(H) \) is \(*\delta\)-homeomorphism.

Theorem 5.6. Every \(*\delta\)-TVS is \(*\delta\)-homogeneous space.

Proof. Let \((X,F,\tau)\) be a \(*\delta\)-TVS. Take \( x, y \in X \) and take \( z = (x - y) + y \). Define a translation map \( T_z : X \rightarrow X \) by \( T_z(x) = x + z \) \( \forall x \in X \). Then \( T_z(x) = y \) for all \( x \in X \). By Theorem 5.3, \( T_z : X \rightarrow X \) is \(*\delta\)-homeomorphism. Hence \( (X,F,\tau) \) is an \(*\delta\)-homogeneous space.
there is a $\ast$-open set $V$ of 0 such that $T_\ast(V) = V + y \subseteq K$.
Since $g$ is $\ast$-continuous at 0 in $X$, $\exists \ast$-open set $U \subseteq X$ containing 0 such that $f(U) \subseteq V$. Since $T_\ast : X \to X$ is $\ast$-homeomorphism, $U + x$ is $\ast$-open set containing $x$. Then $f(U + x) = f(U) + f(x) = f(U) + y \subseteq V + y \subseteq K$. Therefore $g$ is $\ast$-continuous at $x \in X$ and hence on $X$.

**Theorem 6.5.** Let $(X,F)$ be an $\ast$-TVS . The scalar multiple of $\ast$-compact set is $\ast$-compact.

**Proof.** If $\lambda = 0$, we are nothing to prove. Assume that $\lambda$ is non-zero. Let $A$ be a $\ast$-compact subset of $X$ and let $\{U_\alpha : \alpha \in I\}$ be a $\ast$-open cover of $\lambda A$ for some non-zero $\lambda \in F$, then $\lambda A \subseteq \bigcup_{\alpha \in I} U_\alpha$. Then $A \subseteq \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} \left( \frac{1}{\lambda} U_\alpha \right)$. Since $U_\alpha$ is $\ast$-open subset of $\ast$-topological vector space $(X,F)$, $\left( \frac{1}{\lambda} U_\alpha \right)$ is $\ast$-open subset of $X$ for each $\alpha \in I$. Since $A$ is $\ast$-compact, there exists a finite subset $I_0$ of $I$ such that $A \subseteq \bigcup_{\alpha \in I_0} \left( \frac{1}{\lambda} U_\alpha \right)$. This implies that $\lambda A \subseteq \bigcup_{\alpha \in I_0} U_\alpha$. Thus $\lambda A$ is $\ast$-compact in $(X,F)$.

**Theorem 6.6.** Let $(X,F)$ be an $\ast$-TVS . If $K$ is a $\ast$-compact set of $X$ and $G$ is $\ast$-closed subset of $X$ such that $K \cap G = \emptyset$, then $\exists \emptyset$ a $\ast$-open set $U$ containing 0 such that $(K + U) \cap (G + U) = \emptyset$.

**Proof.** If $K = \emptyset$, then the proof is trivial. Otherwise, let $0 = x \in K$, where $K$ is $\ast$-compact. Given that $G$ is $\ast$-closed set. So $G$ is an $\ast$-open subset of $X$ containing 0 = x. Since the addition mapping is $\ast$-continuous and 0 = 0 + 0, therefore there is an $\ast$-open set $U$ containing 0 satisfy $3U = U + U + U \subseteq G$. Define $U_1 = U \cap (-U)$ which is $\ast$-open set, symmetric and $3U_1 = U_1 + U_1 + U_1 \subseteq G^c$. Hence $\{x + x + x, x \in U_1\} \cap G = \emptyset$. Since $U_1$ is symmetric, $(x + U_1 + U_1) \cap (G + U_1) = \emptyset$. By hypothesis, for each $x \in K$ and $K$ is $\ast$-compact, then by the above argument, we have a symmetric $\ast$-open set $V_x$ such that $(x + 2V_x) \cap (G + V_x) = \emptyset$. The sets $\{V_x : x \in K\}$ are a $\ast$-open that covers $K$ and hence $K$ is $\ast$-compact, for finitely number of points $x_j \in K$, $i = 1,2,\ldots,n$, we have $K \subseteq \bigcup_{i=1,2,\ldots,n} (x_j + V_{x_j})$. Define the $\ast$-open set containing 0 by $V = \bigcap_{i=1,2,\ldots,n} V_{x_i}$. Therefore $(K + V) \cap (G + V) \subseteq \bigcup_{i=1,2,\ldots,n} (x_i + V_i + V) \cap (G + V) \subseteq \bigcup_{i=1,2,\ldots,n} (x_i + 2V_i) \cap (G + V_i) = \emptyset$. Hence $(K + U) \cap (G + U) = \emptyset$.

**Lemma 6.7.** Let $(X,F)$ be a $\ast$-TVS , let $U$ be $\ast$-open subset of $X$. If $A$ is any subset of $X$ such that $U \cap A = \emptyset$ then $U \cap cl_\ast(A) = \emptyset$.

**Proof.** Suppose $U \cap cl_\ast(A) \neq \emptyset$. Let $x \in U \cap cl_\ast(A) = \emptyset$. Then $x \in cl_\ast(A)$ and $x \in U$. Since $U$ is $\ast$-closed subset of $X$, $X - U$ is $\ast$-closed subset that contain A. Therefore $cl_\ast(A) \subseteq X - U$, so $x \notin cl_\ast(A)$ which implies a contradiction. Hence $U \cap cl_\ast(A) = \emptyset$.

**Corollary 6.8.** Let $(X,F)$ be $\ast$-TVS . If $\ast$-closed set $G$ and $\ast$-compact set $K$ are disjoint then there is $\ast$-open set $U$ containing 0 such that $cl_\ast(K + U) \cap (G + U) = \emptyset$.

**Proof.** Given that $G$ is $\ast$-closed and $K$ is $\ast$-compact. Let $\lambda > 0$ be Theorem 6.6, there exists $\ast$-open subset of $G$ containing 0 satisfy $(K + U) \cap (G + U) = \emptyset$. The set $G + U = \{y + U : y \in G\}$ is an $\ast$-open set then by Lemma 6.7, $cl_\ast(K + U) \cap (G + U) = \emptyset$.

**Definition 6.1.** A Topological space $X$ is said to be $\ast$-Hausdorff if for every $x \neq y \in X$, there exists a $\ast$-open sets $U_x , V_y$ such that $x \in U_x, y \in V_y$ and $U_x \cap V_y = \emptyset$.

**Definition 6.2.** A Topological space $X$ is called $\ast$-Compact if every cover of $X$ by $\ast$-open sets has finite subcover. A subset $A$ of $X$ is said to be $\ast$-compact if every cover of $A$ by $\ast$-open sets of $X$ has a finite subcover.

**Theorem 6.3.** Every $\ast$-TVS $(X,F)$ is $\ast$-Hausdorff space.

**Proof.** Let $a \in X, a \neq 0$. Since every singleton set in a $\ast$-TVS is $\ast$-closed, $\{a\}$ is $\ast$-closed in $X$. Then $\{a\} = X \setminus \{a\} = U$ (say) is $\ast$-open set containing 0. By Result 2.7, $\exists$ a symmetric $\ast$-open set $V$ containing 0 such that $V + V \subseteq U$. Then by Result 2.8, $a + V = a - V$ is $\ast$-open set. If $V \cap (a - V)$ $\neq \emptyset$, then take $y \in V \cap (a - V)$, $y \in a - V = y = a - x$ for some $x \in V \Rightarrow x + y = a = a \Rightarrow x \in V + V$ as $x,y \in V \Rightarrow a \in U$ which is a contradiction. Therefore $V \cap (a - V) = \emptyset$. Hence the points 0 and $a \neq 0$ are separated by $\ast$-open sets in $X$. Thus $(X,F)$ is $\ast$-Hausdorff space.

**Theorem 6.4.** Let $A$ be $\ast$-compact set in a $\ast$-TVS $(X,F)$. Then $x + A$ is compact $\forall x \in X$.

**Proof.** Let $A$ be $\ast$-compact subset of $\ast$-TVS $X$. Let $\{U_\alpha : \alpha \in I\}$ be a $\ast$-open cover for $x + A$. Then $x + A \subseteq \bigcup_{\alpha \in I} U_\alpha$ which implies that $A \subseteq (-x) + \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (-x + U_\alpha)$. Since $U_\alpha$ is $\ast$-open subset of $\ast$-topological vector space, $(-x + U_\alpha)$ is also $\ast$-open subset of $X$ for each $x \in X$. Since $A$ is $\ast$-compact, there exists a finite subset $I_0$ of $I$ such that $A \subseteq \bigcup_{\alpha \in I_0} (-x + U_\alpha)$. This implies that $x + A \subseteq \bigcup_{\alpha \in I_0} U_\alpha$. Thus $x + A$ is compact.
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