Dominating weakly connected set dominating bridge independent graphs

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Abstract
A γwcsd set S of a connected graph G is a dominating weakly connected set dominating (wcsd) set of G with minimum cardinality. A connected graph G is a γwcsd - excellent if each vertex u of G is in some γwcsd set of G. A graph G is a γwcsd - flexible if to each vertex u of G, there is a γwcsd set not containing u. A wcsd set S of G is wcsd - bridge independent set of G if induced graph of S contains no bridge of G. The minimum cardinality of a wcsd - bridge independent dominating set of G is wcsd - bridge independent dominating number of G and is denoted χwcsd bi(G). A graph G is χwcsd - excellent if every vertex u of G is contained in some χwcsd - set of G. In this paper we have proved that (i) every graph G is an induced sub graph of some χwcsd - excellent, γwcsd - excellent and γwcsd - flexible graph H with γwcsd(G) ≤ γwcsd(H) ≤ χwcsd bi(H) ≤ γwcsd(G) + 1 (ii) Every γwcsd - excellent & γwcsd - flexible is γwcsd bi - excellent and further χwcsd bi = γwcsd (iii) A necessary and sufficient condition under which the graph G = (G1 ∪ G2) + uv where G1 and G2 are disjoint γwcsd excellent graphs and u ∈ V(G1) & v ∈ V(G2) is γwcsd excellent.

Keywords
wcsd - sets, γwcsd - sets, γwcsd - excellent graphs, γwcsd - flexible graphs, χwcsd bi - sets and χwcsd bi - excellent graphs.

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1. Introduction

Sampath Kumar and Pushpa Latha [2] have defined set domination in graphs. Hedetniemi et. al. [4] have defined weakly connected domination in graphs. We defined the concept of weakly connected set dominating sets(wcs), dominating weakly connected set dominating sets(wcsd) and elucidate some results in our earlier paper. We extend these to new class of wcsd-bridge independent graphs and its excellent graphs.
**Definition 3.3.** A connected graph $G$ is a $\gamma_{\text{wcsd}}$ flexible if to each vertex $u$ of $G$, there is a $\gamma_{\text{wcsd}}$ set not containing $u$.

**Definition 3.4.** A wcsd set $S$ of $G$ is wcsd - bridge independent set of $G$ if induced graph of $S$ contains no bridge of $G$. The minimum cardinality of a wcsd - bridge independent dominating set of $G$ is wcsd - bridge independent dominating number of $G$ and is denoted $\gamma^bi_{\text{wcsd}}(G)$.

**Definition 3.5.** A connected graph $G$ is $\gamma^bi_{\text{wcsd}}$ - excellent if every vertex $u$ of $G$ is contained in some $\gamma^bi_{\text{wcsd}}$ - set of $G$.

**Definition 3.6.** A vertex $u\in V(G)$ is called a $\gamma_{\text{wcsd}}$ level vertex of $G$ if $\gamma_{\text{wcsd}}(G-v) = \gamma_{\text{wcsd}}(G)$.

**Definition 3.7.** A vertex $u\in V(G)$ is called a $\gamma_{\text{wcsd}}$ non level vertex of $G$ if $\gamma_{\text{wcsd}}(G-v) = \gamma_{\text{wcsd}}(G) - 1$.

**Observation 3.8.** Every $\gamma_{\text{wcsd}}$ - excellent graph need not be $\gamma_{\text{wcsd}}$ - flexible.

**Example**

\[
S_1 = \{u,a_2,b_3\}, S_2 = \{u,a_3,b_3\}, S_3 = \{u,a_3,b_1\},
S_4 = \{u,a_4,b_2\} \text{ are } \gamma_{\text{wcsd}} - \text{ sets. There is no } \gamma_{\text{wcsd}} - \text{ set without } u. \text{ Therefore it is not } \gamma_{\text{wcsd}} - \text{ flexible.}
\]

**Observation 3.9.** There exists graphs with $\gamma_{\text{wcsd}}$ - excellent and $\gamma^bi_{\text{wcsd}}$ - excellent. Any path with even vertices is both $\gamma_{\text{wcsd}}$ - excellent and $\gamma^bi_{\text{wcsd}}$ - excellent.

**Example**

\[
S_1 = \{2,4\} \text{ is a } \gamma_{\text{wcsd}} \text{ set and induced graph of } S_1 \text{ has no bridge.}
S_2 = \{1,3\} \text{ is a } \gamma_{\text{wcsd}} \text{ set and induced graph of } S_2 \text{ has no bridge.}
\]

**Theorem 3.1.** Every connected graph $G$ of order $n$ is an induced sub graph of $\gamma^bi_{\text{wcsd}}$ - excellent, $\gamma_{\text{wcsd}}$ - flexible graph $H$ of order $n + \gamma_{\text{wcsd}}(G) + 1$ and further $\gamma_{\text{wcsd}}(G) \leq \gamma^bi_{\text{wcsd}}(H) \leq \gamma^bi_{\text{wcsd}}(H) + \gamma_{\text{wcsd}}(G) + 1$.

**Proof.** Let $G$ be a connected graph of order $n$

Let $S = \{v_1, v_2, v_3, \ldots, v_{m-1}, v_m\}$ be a $\gamma_{\text{wcsd}}$ set of $G$ construction of graph $H$ is as follows

\[
V(H) = V(G) \cup \{u_1, u_2, u_3, \ldots, u_{m-1}, u_m, w\}
E(H) = E(G) \cup \{viu_i \mid i = 1, 2, \ldots, m\}
\cup \{vw \mid \forall v \in V - S\}
D = \{v_1, v_2, v_3, \ldots, v_{m-1}, v_m, w\}
D_0 = \{u_1, u_2, u_3, \ldots, u_{m-1}, u_m, w\}
D_i = \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{m-1}, v_m, u_i\}
\forall i = 1, 2, \ldots, m
D_v = \{v_1, v_2, v_3, \ldots, v_{m-1}, v_m, v\}
\forall v \in V - S
\]

are $\gamma_{\text{wcsd}}$ sets of $H$ and for every vertex $u$ in $H$ there exists at least one $\gamma_{\text{wcsd}}$ set of $H$ such that $u$ is not a member of that set. Therefore $H$ is $\gamma_{\text{wcsd}}$ - flexible of order $n + \gamma_{\text{wcsd}}(G) + 1$.

Induced sub graph of these sets contains no bridge of $H$. Therefore $H$ is $\gamma^bi_{\text{wcsd}}$ - excellent graph.

Thus $G$ is an induced sub graph of $H$ such that $\gamma_{\text{wcsd}}(G) \leq \gamma_{\text{wcsd}}(H) \leq \gamma^bi_{\text{wcsd}}(H) \leq \gamma^bi_{\text{wcsd}}(H) + 1$.

**Theorem 3.2.** If $G$ is $\gamma_{\text{wcsd}}$ - excellent and $\gamma_{\text{wcsd}}$ - flexible then $G$ is $\gamma^bi_{\text{wcsd}}$ - excellent and further $\gamma^bi_{\text{wcsd}}(G) = \gamma_{\text{wcsd}}(G)$.

**Proof.** Let $G$ be a connected $\gamma_{\text{wcsd}}$ - excellent and $\gamma_{\text{wcsd}}$ - flexible graph. Let $u \in V(G)$

Since $G$ is $\gamma_{\text{wcsd}}$ - excellent, there exists a $\gamma_{\text{wcsd}}$ - set $S$ of $G$ such that $u \in S$.

Define $d(e) = \min \{d(u,a), d(u,b)/e = ab \text{ is a bridge of } G\}$.

Labeling the bridges of $G$ as $e_1$, $e_2$, ..., $e_{k-1}$, $e_k$ such that $d(e_i) \leq d(e_j)$ whenever $i < j$.

If $<S>$ contains no bridge $e_i$, then $S$ is a $\gamma^bi_{\text{wcsd}}$ - set containing $u$.

Suppose $S$ contains bridge $e_i$ for some $i$ such that $e_1$, $e_2$, ..., $e_{i-1} \notin S$.

Let $e_i = ab$.

Then $G-e_i$ has components $G_1$ & $G_2$ such that $G_1$ contains $a$ & $u$ and $G_2$ contains $b$.

Since $\gamma_{\text{wcsd}}$ - flexible there is a $\gamma_{\text{wcsd}}$ - set $S_1$ of $G$ not containing $b$.

Let $D = (S \cap G_1) \cup (S_1 \cap G_2)$.

Since $S$ and $S_1$ are $\gamma_{\text{wcsd}}$ sets of $G$, $S \cap G_1$ & $S_1 \cap G_2$ are $\gamma_{\text{wcsd}}$ sets of $G_1$ & $G_2$ respectively.

Thus $D$ is $\gamma_{\text{wcsd}}$ set of $G$.

As $e_1$, $e_2$, ..., $e_{i-1} \notin S \supset$ and $e_i \in S \supset$ & $e_1$, $e_2$, ..., $e_{i-1}$ in $G_1$ by our labeling procedure.

Then $e_1$, $e_2$, ..., $e_{i-1}$, $e_i \notin D \supset$.

Proceeding like this for another bridge $e_j$ we get a $\gamma_{\text{wcsd}}$ set $D'$ such that $e_j \notin D' \supset \forall j$.

$D'$ is a $\gamma^bi_{\text{wcsd}}$ - set containing arbitrary $u$.

$G$ is $\gamma^bi_{\text{wcsd}}$ - excellent and $\gamma^bi_{\text{wcsd}}(G) = \gamma_{\text{wcsd}}(G)$.

**Theorem 3.3.** Let $G_1$ and $G_2$ be two $\gamma_{\text{wcsd}}$ - excellent graphs. Let $H = G_1 \cup G_2 + e$ where $e = u_1u_2,u_1 \in G_1,u_2 \in G_2$ then $H$ is $\gamma_{\text{wcsd}}$ - excellent if and only if either $i)$ $u_i$ is a $\gamma_{\text{wcsd}}$ - level vertex of $G_i$, $i = 1, 2$ ii) for each $i$, $u_i$ is...
Assume that which contains $u$. Assume that at least one of $u_i$ belongs to $A_w$ of $G_i$ such that $w, u_i \in A_w$.

**Proof.** Let $G_1$ and $G_2$ be two $\gamma_{wcd}$ - excellent graphs. Let $H = G_1 \cup G_2 + e$ where $e = u_1u_2, u_1 \in G_1, u_2 \in G_2$.

**Case 1:**
Assume that $u_i$ is a $\gamma_{wcd}$ - level vertex of $G_1 i = 1, 2$.
Let $S_1, S_2$ be $\gamma_{wcd}$ - set of $G_1$ and $G_2$ respectively. Then $S_1 \cup S_2$ is a $wcd$ set of $H$

$$\gamma_{wcd}(H) \leq \gamma_{wcd}(G_1) + \gamma_{wcd}(G_2)$$

If $D$ is any $\gamma_{wcd}$ set of $H$ then $D \cap G_1 \subseteq V(G_1)$ and $D \cap G_2 \subseteq V(G_2)$ such that $D \cap G_1$ is $wcd$ set of $G_1 - u_1$ and $D \cap G_2$ is a $wcd$ set of $G_2 - u_2$.

$$\gamma_{wcd}(H) = |D| = |D \cap G_1| + |D \cap G_2|$$
$$\gamma_{wcd}(H) \geq \gamma_{wcd}(G_1 - u_1) + \gamma_{wcd}(G_2 - u_2)$$
$$\gamma_{wcd}(H) \geq \gamma_{wcd}(G_1) + \gamma_{wcd}(G_2)$$

From 1 & 2

$$\gamma_{wcd}(H) = \gamma_{wcd}(G_1) + \gamma_{wcd}(G_2)$$

Let $w \in H$. Then $w \in V(G_i)$ for some $i = \{1, 2\}$.
Let $i \neq j \in \{1, 2\}$
As $G_i$ is a $\gamma_{wcd}$ excellent, there exists a $\gamma_{wcd}$ set $S_i$ of $G_i$ which contains $w$.
Let $S_j$ be any $\gamma_{wcd}$ set of $G_j$.
Then $S_i \cup S_j$ is a $\gamma_{wcd}$ set of $H$ containing $w$.
Thus $H$ is $\gamma_{wcd}$ excellent.

**Case 2:**
Assume that at least one of $u_i$ is a $\gamma_{wcd}$ - non level vertex of $G_i$.
Let $u_1$ be non level vertex of $G_1$. Level set of $u_1$ be $S_1$ and $S_2$ be $\gamma_{wcd}$ set of $G_2$ such that $u_1 \in S_2$.
Implies $S_1 \cup S_2$ is a $wcd$ set of $H$.
Hence

$$\gamma_{wcd}(H) \leq |S_1 \cup S_2| = |S_1| + |S_2|$$

$$\gamma_{wcd}(H) \leq \gamma_{wcd}(G_1) - 1 + \gamma_{wcd}(G_2)$$

Let $A$ be any $\gamma_{wcd}$ set of $H$ and let $A_i = A \cap G_i$.
Then $A_i$ is a $wcd$ set of $G_i - u_i, i = 1, 2$.
And $A_i \cap N(u_i) \neq \emptyset$ for at least one $i$.
So $|A| = |A_1| + |A_2|$

$$|A| \geq \gamma_{wcd}(G_i - u_i) + \gamma_{wcd}(G_j), \quad i \neq j \in \{1, 2\}$$

$$|A| \geq \gamma_{wcd}(G_1) - 1 + \gamma_{wcd}(G_2)$$

$$\gamma_{wcd}(H) \geq \gamma_{wcd}(G_1) - 1 + \gamma_{wcd}(G_2)$$

3 and 4 implies

$$\gamma_{wcd}(H) = \gamma_{wcd}(G_1) - 1 + \gamma_{wcd}(G_2)$$

Sub Case:

Assume that for each $i, u_i$ is a $\gamma_{wcd}$ non level vertex of $G_i$.
For any $w \neq u_i \in G_i$ either $w$ belong to level set of $u_i$ or there is a $\gamma_{wcd}$ set $A_w$ of $G_i$.
Such that $w \& u_i$ both in $A_w$.
We claim that $H$ is a $\gamma_{wcd}$ - excellent.
Let $w \in H$.
If $w$ belongs to some level set $S_1$ of $u_1$ of $G_1$, for any $\gamma_{wcd}$ set $B$ of $G_2$ which contains $u_2$ then $S_1 \cup B$ is a $wcd$ set of $H$ and

$$|S_1 \cup B| = \gamma(G_1) + \gamma(G_2) - 1 = \gamma_{wcd}(H).$$

Implies that $S_1 \cup B$ is a $\gamma_{wcd}$ set of $H$ containing $w$.
Thus $H$ is a $\gamma_{wcd}$ - excellent.
If there is a $\gamma_{wcd}$ set $A_w$ of $G_1$ such that $w \& u_1$ both in $A_w$.
We can choose a level set $B$ of $u_2$ of $G_2$ then $A_w \cup B$ is a $wcd$ set of $H$ containing $w$

$$|A_w \cup B| = |A_w| + |B|$$
$$= \gamma_{wcd}(G_1) + \gamma_{wcd}(G_2) - 1$$
$$= \gamma_{wcd}(H)$$

Implies that $A_w \cup B$ is a $\gamma_{wcd}$ set of $H$ containing $w$.
Thus $H$ is a $\gamma_{wcd}$ - excellent.

\[\square\]

### 4. Conclusion

This paper has attempted to establish new class of bridge independent excellent graphs with respect to the parameter dominating weakly connected set domination and enabled to study various properties of such graphs. The future scope of study is to make new class of bridge independent excellent graphs with respect to the parameter weakly connected point set.

### References