Malaya Journal of Matematik, Vol. S, No. 1, 415-423, 2019

https://doi.org/10.26637/MJM0S01/0074

Fuzzy M^*-open sets in $\hat{S}$ostak's fuzzy topological spaces

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Abstract

The aim of this paper is to define ι-fuzzy M^*-open and ι-fuzzy M^*-closed sets in $\hat{S}$ostak's fuzzy topological spaces. Also, ι-fuzzy M^*-interior, ι-fuzzy M^*-closure are introduced and their properties are investigated. Moreover, we investigate the connections between ι-fuzzy open, ι-fuzzy θ-semiopen, ι-fuzzy θ-open, ι-fuzzy δ-semiopen, ι-fuzzy δ-preopen, ι-fuzzy a-open, ι-fuzzy e-open and ι-fuzzy e^*-open in fuzzy topological spaces in the sense of $\hat{S}$ostak.

Keywords

ι-fuzzy M^*-open, ι-fuzzy M^*-closed, ι-fuzzy M^*-interior, ι-fuzzy M^*-closure, ι-fuzzy M-open, ι-fuzzy M-closed, ι-fuzzy M-interior, ι-fuzzy M-closure, ι-fuzzy θ-interior, ι-fuzzy θ-closure, ι-fuzzy θ-semiopen, ι-fuzzy θ-open, ι-fuzzy δ-semiopen, ι-fuzzy δ-preopen, ι-fuzzy a-open, ι-fuzzy e-open and ι-fuzzy e^*-open.

AMS Subject Classification

54A40, 54C05, 03E72.

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Article History: Received **24** November **2018**; Accepted **09** April **2019**

Contents

1. Introduction

Chang's fuzzy topology [4] has been extended by $\hat{S}$ostak [28] in fuzzy topology and different level of growths have been made in [13, 14, 26]. Various authors [2, 3, 5, 10, 12, 21, 23] have developed fuzzy continuity between fuzzy topological space in weaker forms also using the idea of fuzzy semi-open sets [2], fuzzy regular open sets [2], fuzzy preopen sets, fuzzy strongly semiopen sets [3], fuzzy γ-open sets [12], fuzzy δ-semiopen sets [1], fuzzy δ-preopen sets [1], fuzzy semi δ-preopen sets [33] and fuzzy e-open sets [27]. In the sense of Chang [4] fts, Ganguly and Saha [11] developed the idea of fuzzy δ-cluster points in fts. In the sense of $\hat{S}$ostak's fuzzy topological space, Kim and Park [15] developed ι-δ-cluster points and δ-closure operators. The weaker forms of fuzzy semi-preopen sets was developed by Park et al. [19] than any other from of fuzzy semi-open or fuzzy preopen sets. In 2008, the formations of e-open sets, e^*-open sets and a-open sets in topological spaces are due to Erdal Ekici [[8], [9]]. Sobana et al. [29] defined ι-fuzzy e-open and ι-fuzzy e-closed sets in a fuzzy topological space in the sense of $\hat{S}$ostak. Vadivel et al. [31] introduced ι-fuzzy e^*-open and ι-fuzzy e^*-closed sets in a fuzzy topological space in the sense of $\hat{S}$ostak. Velicko [30] in 1968 has developed and analyzed a specific variety of θ-open sets and δ-open sets in characherizations of H-closed topological spaces.

A new type of semi-open set [18] was developed by Levine in 1963, whereas Raychaudhuri and Mukherjee contributed δ-preopen sets [25]. Park [20] in 1997 defined δ-semiopen sets also Caldas [6] developed θ-semi-open sets in 2008. Shafei introduced fuzzy θ-closed [34] and fuzzy θ-open sets in 2006. Maghrabi et al.[22] introduced the notion of M-open sets in topological spaces in the year 2011. Devika et.al [7] developed the concept of M^*-open sets in topological spaces in the year 2016.

This article is a classified analysis of a different class of open set namely ι-fuzzy M^*-open set. In this paper, we identify the idea of ι-fuzzy M^*-open (resp. ι-fuzzy M^*-closed) sets in fuzzy topological spaces in the sense of $\hat{S}$ostak's. Also, we defined ι-fuzzy M^*-interior (resp. ι-fuzzy M^*-closure) and analysed some of their properties. Also the relationships of ι-fuzzy open, ι-fuzzy θ-semiopen, ι-fuzzy θ-open, ι-fuzzy δ-semiopen, ι-fuzzy δ-preopen, ι-fuzzy a-open, ι-fuzzy e-open and ι-fuzzy e^*-open in $\hat{S}$ostak's fuzzy topological spaces are analysed.

2. Preliminaries

Throughout this article, we denote nonempty sets by X, Y etc., $I=[0,\ 1]$ and $I_0=(0,\ 1]$. For $\alpha\in I$, $\overline{\alpha}(x)=\alpha,\quad \forall x\in X$. A fuzzy point x_t for $t\in I_0$ is an element of I^X such that $x_t(y)=\begin{cases} t & \text{if y is equal to x}\\ 0 & \text{if y is not equal to x.}\end{cases}$
Let $Pt(X)$ denote the set of all fuzzy points in X. A fuzzy point $x_t\in\mu$ iff $t<\mu(x)$. $\mu\in I^X$ is quasi-coincident with ν, denoted by $\mu q\nu$, if $\exists\, x\in X$ such that $\mu(x)+\nu(x)>1$.
If μ is not quasi-coincident with ν, we denoted $\mu\bar{q}\nu$. If A is a subset of X, we define the characteristic function χ_A on X by $\chi_A(x)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A.\end{cases}$ All notations and definitions will be standard in the fuzzy set theory.

Lemma 2.1. *[28] Consider X be a nonempty set and $\mu,\ \nu\in I^X$. Then*

(i) $\mu q\nu$ iff there exists $x_t\in\mu$ such that $x_tq\nu$.

(ii) $\mu q\nu$, then $\mu\wedge\nu\neq\overline{0}$.

(iii) $\mu\bar{q}\nu$ iff $\mu\leq\overline{1}-\nu$.

(iv) $\mu\leq\nu$ iff $x_t\in\mu$ implies $x_t\in\nu$ iff $x_tq\mu$ implies $x_tq\nu$ implies $x_t\bar{q}\mu$.

(v) $x_t\bar{q}\bigvee_{i\in\mu}\nu_i$ iff there exists $i_0\in\mu$ such that $x_t\bar{q}\nu_{i_0}$.

Definition 2.2. *[28] A function $\tau: I^X\to I$ is called a fuzzy topology on X if it satisfies the following conditions:*

(1) $\tau(\overline{0})=\tau(\overline{1})=1$,

(2) $\tau(\bigvee_{i\in\Gamma}\nu_i)\geq\bigwedge_{i\in\Gamma}\tau(\nu_i)$, for any $\{\nu_i\}_{i\in\Gamma}\subset I^X$,

(3) $\tau(\nu_1\wedge\nu_2)\geq\tau(\nu_1)\wedge\tau(\nu_2)$, for any $\nu_1,\ \nu_2\in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space or $\hat{S}$ostak's fuzzy topological space or smooth topological space (for short, fts, sfts, sts).

Remark 2.3. *[24] Let $(X,\ \tau)$ be a sfts. Then, for every $\iota\in I_0$, $\tau_r=\{\nu\in I^X:\tau(\nu)\geq\iota\}$ is a Change's fuzzy topology on X.*

Theorem 2.4. *[26] Let $(X,\ \tau)$ be a sfts. Then for each $\mu\in I^X,\ \iota\in I_0$, we define an operator $C_\tau: I^X\times I_0\to I^X$ as follows:*
$C_\tau(\mu,\ \iota)=\bigwedge\{\nu\in I^X:\mu\leq\nu,\ \tau(\overline{1}-\nu)\geq\iota\}$.
For $\mu,\ \nu\in I^X$ and $\iota,s\in I_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\overline{0},\iota)=\overline{0}$,

(2) $\mu\leq C_\tau(\mu,\ \iota)$,

(3) $C_\tau(\mu,\ \iota)\vee C_\tau(\nu,\ \iota)=C_\tau(\mu\vee\nu,\ \iota)$,

(4) $C_\tau(\mu,\ \iota)\leq C_\tau(\mu,\ s)$ if $\iota\leq s$,

(5) $C_\tau(C_\tau(\mu,\ \iota),\ \iota)=C_\tau(\mu,\ \iota)$.

Theorem 2.5. *[26] Let $(X,\ \tau)$ be a sfts. Then for each $\iota\in I_0,\ \mu\in I^X$ we define an operator $I_\tau: I^X\times I_0\to I^X$ as follows:*
$I_\tau(\mu,\ \iota)=\bigvee\{\nu\in I^X:\mu\geq\nu,\ \tau(\nu)\geq\iota\}$.
For $\mu,\ \nu\in I^X$ and $\iota,s\in I_0$, the operator I_τ satisfies the following conditions:

(1) $I_\tau(\overline{1},\iota)=\overline{1}$,

(2) $\mu\geq I_\tau(\mu,\ \iota)$,

(3) $I_\tau(\mu,\ \iota)\wedge I_\tau(\nu,\ \iota)=I_\tau(\mu\wedge\nu,\ \iota)$,

(4) $I_\tau(\mu,\ \iota)\leq I_\tau(\mu,\ s)$ if $s\leq\iota$,

(5) $I_\tau(I_\tau(\mu,\ \iota),\ \iota)=I_\tau(\mu,\ \iota)$,

(6) $I_\tau(\overline{1}-\mu,\iota)=\overline{1}-C_\tau(\mu,\ \iota)$ and $C_\tau(\overline{1}-\mu,\iota)=\overline{1}-I_\tau(\mu,\ \iota)$

Definition 2.6. *[16] Let $(X,\ \tau)$ be a sfts. Then for each $\nu\in I^X$, $x_t\in P_t(X)$ and $\iota\in I_0$, ν is called*

(i) ι-open Q_τ-neighbourhood of x_t if $x_tq\nu$ with $\tau(\nu)\geq\iota$.

(ii) ι-open R_τ-neighbourhood of x_t if $x_tq\nu$ with $\nu=I_\tau(C_\tau(\mu,\ \iota),\ \iota)$.

We denote $Q_\tau(x_t,\ \iota)=\{\nu\in I^X:x_tq\nu,\tau(\nu)\geq\iota\}$, $R_\tau(x_t,\ \iota)=\{\nu\in I^X:x_tq\nu=I_\tau(C_\tau(\mu,\ \iota),\ \iota)\}$.

Definition 2.7. *[16] Let $(X,\ \tau)$ be a sfts. Then for each $\mu\in I^X$, $x_t\in P_t(X)$ and $\iota\in I_0$, x_t is called*

(i) ι-τ cluster point of μ if for every $\nu\in Q_\tau(x_t,\ \iota)$, we have $\nu q\mu$.

(ii) ι-δ cluster point of μ if for every $\nu\in R_\tau(x_t,\ \iota)$, we have $\nu q\mu$.

(iii) An δ-closure operator is a mapping $D_\tau: I^X\times I\to I^X$ defined as follows: $\delta C_\tau(\mu,\ \iota)$ or $D_\tau(\mu,\ \iota)=\bigvee\{x_t\in P_t(X):x_t$ is r-δ-cluster point of $\mu\}$

Definition 2.8. *Let $(X,\ \tau)$ be a sfts. For $\mu,\ \nu\in I^X$ and $\iota\in I_0$, μ is called an*

(i) ι-fuzzy δ-semiopen (resp. ι-fuzzy δ-semiclosed)[29] set if $\mu\leq C_\tau(\delta I_\tau(\mu,\ \iota),\ \iota)$ (resp. $I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota)\leq\mu$).

(ii) ι-fuzzy δ-preopen (resp. ι-fuzzy δ-preclosed)[29] set if $\mu\leq I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota)$ (resp. $C_\tau(\delta I_\tau(\mu,\ \iota),\ \iota)\leq\mu$).

(iii) *ι-fuzzy a-open (resp. ι-fuzzy a-closed)[29] set if* $\mu \leq I_\tau(C_\tau(\delta I_\tau(\mu,\ \iota),\ \iota),\ \iota)$ *(resp.* $C_\tau(I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota),\ \iota) \leq \mu$*).*

(iv) *ι-fuzzy e-open (resp. ι-fuzzy e-closed)[29] set if* $\mu \leq C_\tau(\delta I_\tau(\mu,\ \iota),\ \iota) \vee I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota)$ *(resp.* $C_\tau(\delta I_\tau(\mu,\ \iota),\ \iota) \wedge I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota) \leq \mu$*).*

(v) *ι-fuzzy e^*-open (resp. ι-fuzzy e^*-closed)[31] set if* $\mu \leq C_\tau(I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota),\ \iota)$ *(resp.* $I_\tau(C_\tau(\delta I_\tau(\mu,\ \iota),\ \iota),\ \iota) \leq \mu$*).*

(vi) *ι-fuzzy semiopen (resp. ι-fuzzy semi-closed) [17] set if* $\mu \leq C_\tau(I_\tau(\mu,\ \iota),\ \iota)$ *(resp.* $I_\tau(C_\tau(\mu,\ \iota),\ \iota) \leq \mu$*).*

(vii) *ι-fuzzy M-open (resp. ι-fuzzy M-closed)[32] set if* $\mu \leq C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) \vee I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota)$. *(resp.* $\mu \geq C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) \wedge I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota)$.*)*

Definition 2.9. *[32] Let $(X,\ \tau)$ be a sfts, $\mu,\ \nu \in I^X$ and $\iota \in I_0$, then*

(i) *The ι-fuzzy θ-interior (resp. θ-closure) of μ is*

$$\theta I_\tau(\mu,\ \iota) = \bigvee\{I_\tau(\nu) : \mu \geq \nu,\ \tau(\overline{1}-\nu) \geq \iota\}$$

(resp. $\theta C_\tau(\mu,\ \iota) = \bigwedge\{C_\tau(\nu) : \mu \leq \nu,\ \tau(\nu) \geq \iota\}$.*)*

(ii) *The ι-fuzzy θ-semi-interior (resp. θ-semi-closure) of μ is* $\theta sI_\tau(\mu,\ \iota) = \bigvee\{sI_\tau(\nu) : \mu \geq \nu,\ \nu \text{ is } \iota\text{-fsc}\}$ *(resp.* $\theta sC_\tau(\mu,\ \iota) = \bigwedge\{sC_\tau(\nu) : \mu \leq \nu,\ \nu \text{ is } \iota\text{-fso}\}$.*)*

(iii) *The ι-fuzzy θ-pre-interior (resp. θ-pre-closure) of μ is* $\theta pI_\tau(\mu,\ \iota) = \bigvee\{pI_\tau(\nu) : \mu \geq \nu,\ \nu \text{ is } \iota\text{-fpc}\}$ *(resp.* $\theta pC_\tau(\mu,\ \iota) = \bigwedge\{pC_\tau(\nu) : \mu \leq \nu,\ \nu \text{ is } \iota\text{-fpo}\}$.*)*

Definition 2.10. *[32] Let $(X,\ \tau)$ be a sfts. For $\mu,\ \nu \in I^X$ and $\iota \in I_0$, μ is called an*

(i) *ι-fuzzy θ-open (resp. θ-closed) set if* $\mu = \theta I_\tau(\mu,\ \iota)$ *(resp.* $\mu = \theta C_\tau(\mu,\ \iota)$*).*

(ii) *ι-fuzzy θ-semiopen (resp. θ-semiclosed) set if* $\mu \leq C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$ *(resp.* $I_\tau(\theta C_\tau(\mu,\ \iota),\ \iota) \leq \mu$*).*

(iii) *ι-fuzzy θ-preopen (resp. θ-preclosed) set if* $\mu \leq I_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$ *(resp.* $C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) \leq \mu$*).*

Definition 2.11. *[32] Let $(X,\ \tau)$ be a sfts. Then*

(i) *The union of all ι-fuzzy θ-open, (resp. ι-fuzzy θ-semiopen, ι-fuzzy θ-preopen) sets contained in μ is called the ι-fuzzy θ-interior (resp. ι-fuzzy θ-semi-interior, ι-fuzzy θ-pre-interior) of μ and is denoted by $\theta I_\tau(\mu,\ \iota)$ (resp. $\theta sI_\tau(\mu,\ \iota)$, $\theta pI_\tau(\mu,\ \iota)$).*

(ii) *The intersection of all ι-fuzzy θ-closed, (resp. ι-fuzzy θ-semiclosed, ι-fuzzy θ-preclosed) sets containing μ is called the ι-fuzzy θ-closure (resp. ι-fuzzy θ-semi-closure, ι-fuzzy θ-pre-closure) of μ and is denoted by $\theta C_\tau(\mu,\ \iota)$ (resp. $\theta sC_\tau(\mu,\ \iota)$, $\theta pC_\tau(\mu,\ \iota)$).*

Lemma 2.12. *[32] Let $(X,\ \tau)$ be a sfts, $\mu,\ \nu \in I^X$ and $\iota \in I_0$, then*

(i) *μ is ι-fuzzy θ-open if and only if* $\mu = \theta I_\tau(\mu,\ \iota)$.

(ii) $\theta I_\tau(\mu,\ \iota)$ *is the union of all fuzzy open sets of X whose closures are contained in μ.*

(iii) $\theta I_\tau(\theta I_\tau(\mu,\ \iota)) < \theta I_\tau(\mu,\ \iota)$.

(iv) *For any subset μ of X,* $\mu \leq C_\tau(\mu,\ \iota) \leq \delta C_\tau(\mu,\ \iota) \leq \theta C_\tau(\mu,\ \iota)$ *(resp.* $\theta I_\tau(\mu,\ \iota) \leq \delta I_\tau(\mu,\ \iota) \leq I_\tau(\mu,\ \iota) \leq \mu$.

(v) $\overline{1} - (\theta I_\tau(\mu,\ \iota)) = \theta C_\tau(\overline{1} - \mu,\ \iota)$.

(vi) $\overline{1} - (\theta C_\tau(\mu,\ \iota)) = \theta I_\tau(\overline{1} - \mu,\ \iota)$.

(vii) *If $\mu < \nu$, then* $\theta I_\tau(\mu,\ \iota) < \theta I_\tau(\nu,\ \iota)$.

(viii) $\theta I_\tau(\mu \wedge \nu,\ \iota) = \theta I_\tau(\mu,\ \iota) \wedge \theta I_\tau(\nu,\ \iota)$ $\theta I_\tau(\mu,\ \iota) \vee \theta I_\tau(\nu,\ \iota) < \theta I_\tau(\mu \vee \nu,\ \iota)$.

(ix) $\theta C_\tau(\mu \wedge \nu,\ \iota) = \theta C_\tau(\mu,\ \iota) \wedge \theta C_\tau(\nu,\ \iota)$ *and* $\theta C_\tau(\mu \vee \nu,\ \iota) = \theta C_\tau(\mu,\ \iota) \vee \theta C_\tau(\nu,\ \iota)$.

Proposition 2.13. *[32] Let $(X,\ \tau)$ be a sfts, $\mu \in I^X$ and $\iota \in I_0$, then the following statements hold :*

(i) $\theta sC_\tau(\mu,\ \iota) = \mu \vee I_\tau(\theta C_\tau(\mu,\ \iota),\iota)$, $\theta sI_\tau(\mu,\ \iota) = \mu \wedge C_\tau(\theta I_\tau(\mu,\ \iota),\iota)$.

(ii) $\delta pC_\tau(\mu,\ \iota) = \mu \vee C_\tau(\delta I_\tau(\mu,\ \iota),\iota)$, $\delta pI_\tau(\mu,\ \iota) = \mu \wedge I_\tau(\delta C_\tau(\mu,\ \iota),\iota)$.

(iii) $\overline{1} - (\delta I_\tau(\mu,\ \iota)) = \delta C_\tau(\overline{1} - \mu,\ \iota)$ *and* $\delta I_\tau(\overline{1} - \mu,\ \iota) = \overline{1} - \delta C_\tau(\mu,\ \iota)$

Lemma 2.14. *[32] Let $(X,\ \tau)$ be a sfts. For $\mu \in I^X$ and $\iota \in I_0$, then the following statements are hold.*

(i) $pI_\tau(\delta pC_\tau(\mu,\ \iota),\iota) = \delta pC_\tau(\mu,\iota) \wedge I_\tau(C_\tau(\mu,\ \iota),\iota)$ *and* $pC_\tau(\delta pI_\tau(\mu,\iota),\iota) = \delta pI_\tau(\mu,\iota) \vee C_\tau(I_\tau(\mu,\iota),\iota)$.

(ii) $\theta pI_\tau(\delta pC_\tau(\mu,\iota),\iota) = \delta pC_\tau(\mu,\iota) \wedge I_\tau(\theta C_\tau(\mu,\ \iota),\iota)$ *and* $\theta pC_\tau(\delta pI_\tau(\mu,\iota),\iota) = \delta pI_\tau(\mu,\iota) \vee C_\tau(\theta I_\tau(\mu,\ \iota),\iota)$.

(iii) $\theta sC_\tau(\theta I_\tau(\mu,\ \iota),\iota) = sC_\tau(\theta I_\tau(\mu,\iota),\iota)$ *and* $sC_\tau(\theta I_\tau(\mu,\iota),\iota) = I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota)$.

(iv) $\theta sI_\tau(\theta C_\tau(\mu,\ \iota),\iota) = sI_\tau(\theta C_\tau(\mu,\iota),\iota)$ *and* $sI_\tau(\theta C_\tau(\mu,\iota),\iota) = C_\tau(I_\tau(\theta C_\tau(\mu,\ \iota),\iota),\iota)$.

Definition 2.15. *[32] Let $(X,\ \tau)$ be a sfts, $\mu,\ \nu \in I^X$ and $\iota \in I_0$, then* *(i)* $MI_\tau(\mu,\, \iota) = \bigvee\{\nu \in I^X : \mu \geq \nu,\ \nu$ *is a ι-fMo set* $\}$ *is called the ι-fuzzy M-interior of μ.*

(ii) $MC_\tau(\mu,\, \iota) = \bigwedge\{\nu \in I^X : \mu \leq \nu,\ \nu$ *is a ι-fMc set* $\}$ *is called the ι-fuzzy M-closure of μ.*

Proposition 2.16. *[32] Let $(X,\, \tau)$ be a sfts, $\mu \in I^X$ and $\iota \in I_0$, then*

(i) Every ι-fuzzy θ-semiopen (resp. ι-fuzzy δ-preopen) set is ι-fuzzy M-open.

(ii) Every ι-fuzzy M-open set is ι-fuzzy e-open.

3. ι-fuzzy M^*-open sets and ι-fuzzy M^*-closed sets

Definition 3.1. *Let $(X,\, \tau)$ be a sfts. For $\mu \in I^X$ and $\iota \in I_0$, μ is called an ι-fuzzy*

(i) M^*-open set if $\mu \leq I_\tau(C_\tau(\theta I_\tau(\mu,\, \iota),\, \iota),\, \iota)$.

(ii) M^*-closed set if $\mu \geq C_\tau(I_\tau(\theta C_\tau(\mu,\, \iota),\, \iota),\, \iota)$.

The collection of all ι-fuzzy M^*-open (resp. ι-fuzzy M^*-closed) sets will be denoted by ι-fM^*o (resp. ι-fM^*c) respectively.

Definition 3.2. *Let $(X,\, \tau)$ be a sfts, $\mu,\, \nu \in I^X$ and $\iota \in I_0$,*

(i) $M^*I_\tau(\mu,\, \iota) = \bigvee\{\nu \in I^X : \mu \geq \nu,\ \nu$ *is a ι-fM^*o set* $\}$ *is called the ι-fuzzy M^*-interior of μ.*

(ii) $M^*C_\tau(\mu,\, \iota) = \bigwedge\{\nu \in I^X : \mu \leq \nu,\ \nu$ *is a ι-fM^*c set* $\}$ *is called the ι-fuzzy M^*-closure of μ.*

Obviously, $M^*C_\tau(\mu,\, \iota)$ is the smallest ι-fM^*c set which contains μ and $M^*I_\tau(\mu,\, \iota)$ is the largest ι-fM^*o set which is contained in μ. Also $M^*C_\tau(\mu,\, \iota) = (\mu,\, \iota)$ for any ι-fM^*c set μ and $M^*I_\tau(\mu,\, \iota) = (\mu,\, \iota)$ for any ι-fM^*o set μ.

Proposition 3.3. *The following are equivalent for a subset μ of a sfts $(X,\, \tau)$.*

(i) Every ι-fuzzy θ-open set is an ι-fuzzy M^*-open set.

(ii) Every ι-fuzzy M^*-open set is an ι-fuzzy θ-semiopen set.

(iii) Every ι-fuzzy M^*-open set is an ι-fuzzy M-open set.

Proof. (i) Let μ be an ι-fuzzy θ-open set. Then $\mu = \theta I_\tau(\mu,\, \iota)$ and by Lemma 2.12(iv), $\theta I_\tau(\mu,\, \iota) \leq I_\tau(\mu,\, \iota) \leq \mu$. Hence $\mu = I_\tau(\mu,\, \iota)$. Since $\mu = \theta I_\tau(\mu,\, \iota) \leq C_\tau(\theta I_\tau(\mu,\, \iota),\, \iota)$, then $\mu = I_\tau(\mu,\, \iota) \leq I_\tau(C_\tau(\theta I_\tau(\mu,\, \iota),\, \iota),\, \iota)$. Thus μ is ι-fM^*o set.

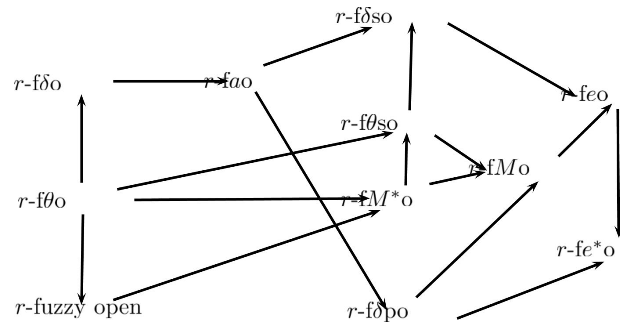
(ii) Obvious from the definition.

(iii) Let μ be ι-fM^*o set. Then

$$\begin{aligned}\mu &\leq I_\tau(C_\tau(\theta I_\tau(\mu,\, \iota),\, \iota),\, \iota)\\ &\leq C_\tau(\theta I_\tau(\mu,\, \iota),\, \iota)\\ &\leq C_\tau(\theta I_\tau(\mu,\, \iota),\, \iota) \vee I_\tau(\delta C_\tau(\mu,\, \iota),\, \iota).\end{aligned}$$

Hence μ is an ι-fM^*o set. □

Remark 3.4. *From the above Definitions and Proposition 3.3, it is clear that the following implications are true for a subset μ of a sfts X and $\iota \in I_0$.*



where ι-fθso, ι-fθsc, ι-fθo, ι-fθc, ι-fδso, ι-fδsc, ι-fδpo, ι-fδpc, ι-fao, ι-fac, ι-fM^*o, ι-fM^*c, ι-fMo, ι-fMc, ι-feo, ι-fec, ι-fe^*o and ι-fe^*c, are abbreviated by ι-fuzzy θ-semiopen, ι-fuzzy θ-semiclosed, ι-fuzzy θ-open, ι-fuzzy θ-closed, ι-fuzzy δ-semiopen, ι-fuzzy δ-semiclosed, ι-fuzzy δ-preopen, ι-fuzzy δ-preclosed, ι-fuzzy a-open, ι-fuzzy a-closed, ι-fuzzy M^*-open, ι-fuzzy M^*-closed, ι-fuzzy M-open, ι-fuzzy M-closed, ι-fuzzy e-open, ι-fuzzy e-closed, ι-fuzzy e^*-open and ι-fuzzy e^*-closed respectively.

From [32], it is clear that every ι-fδpo is ι-fMo and every ι-fθso is ι-fMo. Also, it is clear that every ι-fMo set is ι-feo set and ι-fe*o set. Also, every ι-fθo, ι-fδo, ι-fao set is ι-fMo set. The converses need not be true in general. They were shown by the examples are in the paper [32] and therein.

Example 3.5. *Let μ and ν be fuzzy subsets of $X = \{a,\, b,\, c\}$ defined as follows:*
$\mu(a) = 0.3,\ \mu(b) = 0.5,\ \mu(c) = 0.5$;
$\nu(a) = 0.5,\ \nu(b) = 0.5,\ \nu(c) = 0.5$.
Then $\tau : I^X \to I$ defined as

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1},\\ \frac{1}{2}, & \text{if } \mu = \mu,\ \nu,\\ 0, & \text{otherwise,}\end{cases}$$

is a sfts on X. For ι=$\frac{1}{2}$, then ν is $\frac{1}{2}$-fM^*o and $\frac{1}{2}$-fθo set.

Example 3.6. *Let μ, ν and ω be fuzzy subsets of $X = \{a,\, b,\, c\}$ defined as follows:*
$\mu(a) = 0.3,\ \mu(b) = 0.4,\ \mu(c) = 0.5$;
$\nu(a) = 0.6,\ \nu(b) = 0.5,\ \nu(c) = 0.5$;
$\omega(a) = 0.7,\ \omega(b) = 0.6,\ \omega(c) = 0.5$.
Then $\tau : I^X \to I$ defined as

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1},\\ \frac{1}{2}, & \text{if } \mu = \mu,\ \nu,\\ 0, & \text{otherwise,}\end{cases}$$

is a sfts on X. For ι=$\frac{1}{2}$, then ω is $\frac{1}{2}$-fθso set but ω is not $\frac{1}{2}$-fM^*o set.

Example 3.7. *Let μ and ν be fuzzy subsets of $X = \{a, b, c\}$ defined as follows:*
$\mu(a) = 0.1$, $\mu(b) = 0.1$, $\mu(c) = 0.1$;
$\nu(a) = 0.9$, $\nu(b) = 0.9$, $\nu(c) = 0.9$.
Then $\tau : I^X \to I$ defined as

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \mu = \mu, \\ 0, & \text{otherwise}, \end{cases}$$

is a smooth fuzzy topology on X. For ι=$\frac{1}{2}$, then ν is $\frac{1}{2}$-fMoset but ν is not $\frac{1}{2}$-fM^*o set.

Theorem 3.8. *Let (X, τ) be a sfts and $\iota \in I_o$.*

(i) Arbitrary union of ι-fM^*o sets is an ι-fM^*o set.

(ii) Arbitrary intersection of ι-fM^*c sets is an ι-fM^*c set.

Proof. (i) Let $\{\mu_\alpha : \alpha \in \Gamma\}$ be a family of ι-fM^*o sets. For each $\alpha \in \Gamma$,

$$\begin{aligned} \mu_\alpha &\le I_\tau(C_\tau(\theta I_\tau(\mu_\alpha, \iota), \iota), \iota). \\ \bigvee_{\alpha\in\Gamma} \mu_\alpha &\le \bigvee_{\alpha\in\Gamma} I_\tau(C_\tau(\theta I_\tau(\mu_\alpha, \iota), \iota), \iota). \\ &\le I_\tau(C_\tau(\theta I_\tau(\vee\mu_\alpha, \iota), \iota), \iota). \end{aligned}$$

(ii) Similar to the proof of (i). □

Lemma 3.9. *For a sfts (X, τ), then the family of all ι-fuzzy M^*-open sets of X forms a smooth topology denoted by $\tau_{M_r^*}$ for X.*

Proof. It is obvious that $\overline{0}$ and $\overline{1}$ are in fM^*o sets of X and from Theorem 3.8, we have arbitrary union of ι-fM^*o sets is an ι-fM^*o set.

Let μ and ν be ι-fM^*o sets. Then

$$\mu \le I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota)$$

and

$$\nu \le I_\tau(C_\tau(\theta I_\tau(\nu, \iota), \iota), \iota).$$

Hence $\mu \wedge \nu$

$$\begin{aligned} &\le I_\tau(C_\tau(\theta I_\tau(\mu, \iota),\iota),\iota) \wedge I_\tau(C_\tau(\theta I_\tau(\nu, \iota),\iota),\iota) \\ &\le I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota) \wedge C_\tau(\theta I_\tau(\nu, \iota), \iota),\iota) \\ &\le I_\tau(C_\tau(\theta I_\tau(\mu, \iota) \wedge \theta I_\tau(\nu, \iota), \iota), \iota) \\ &\le I_\tau(C_\tau(\theta I_\tau(\mu \wedge \nu, \iota), \iota), \iota) \end{aligned}$$

Hence the finite intersection of ι-fM^*o sets is ι-fM^*o and hence $\tau_{M_r^*}$ is a smooth topology for X. □

Theorem 3.10. *The following hold for a subset μ of a fts X.*

(i) μ is ι-fM^*o $\Leftrightarrow \mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota)$.

(ii) μ is ι-fM^*c $\Leftrightarrow \mu = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota)$.

(iii) $M^*I_\tau(\mu, \iota) = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota)$.

(iv) $M^*C_\tau(\mu, \iota) = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota)$.

Proof. (i) Let μ be ι-fM^*o. Then

$$\mu \le I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota).$$

We obtain $\mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota)$.
Conversely, let $\mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota)$.
We have

$$\begin{aligned} \mu &= \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota). \\ &\le I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota). \end{aligned}$$

Hence μ is ι-fM^*o.
(ii) Let μ be ι-fM^*c. Then $\mu \ge C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota)$.
We obtain $\mu = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota)$.
Conversely, let $\mu = (\mu, \iota) \vee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota)$.
We have

$$\begin{aligned} \mu &= \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota). \\ &\ge C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota). \end{aligned}$$

Hence μ is ι-fM^*c.
(iii) Since $M^*I_\tau(\mu, \iota)$ is ι-fM^*o, we have,

$$\begin{aligned} I_\tau(C_\tau(\theta I_\tau(\mu, \iota),\iota),\iota) &\ge I_\tau(C_\tau(\theta I_\tau(M^*I_\tau(\mu, \iota),\iota),\iota), \iota) \\ &\ge M^*I_\tau(\mu, \iota). \end{aligned}$$

Hence,
$\mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota) \ge M^*I_\tau(\mu, \iota)$.
On the other way, since
$I_\tau(C_\tau(\theta I_\tau(\mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota), \iota), \iota), \iota)$

$$\begin{aligned} &\ge I_\tau(C_\tau(\theta I_\tau(\mu \wedge \theta I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota), \iota), \iota), \iota) \\ &\ge I_\tau(C_\tau(\theta I_\tau(\mu, \iota) \wedge \theta I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota), \iota), \iota) \\ &\ge I_\tau(C_\tau(\theta I_\tau(\mu, \iota) \wedge \theta I_\tau(\theta I_\tau(\mu, \iota), \iota), \iota), \iota) \\ &= I_\tau(C_\tau(\theta I_\tau(\mu \wedge \theta I_\tau(\mu, \iota), \iota), \iota), \iota) \\ &\ge \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota), \end{aligned}$$

This implies that $\mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, \iota), \iota), \iota)$ is ι-fM^*o contained in μ. Hence
$M^*I_\tau(\mu, \iota) \ge \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota)$.
Thus, we obtain

$$M^*I_\tau(\mu, \iota) = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota).$$

(iv) Since $M^*C_\tau(\mu, \iota)$ is ι-fM^*c, we have,
$C_\tau(I_\tau(\theta C_\tau(\mu, \iota),\iota),\iota)$

$$\begin{aligned} &= C_\tau(I_\tau(\theta C_\tau(M^*C_\tau(\mu, \iota),\iota),\iota), \iota) \\ &\le M^*C_\tau(\mu, \iota). \end{aligned}$$

Hence
$\mu \bigvee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota) \le M^*C_\tau(\mu, \iota)$.
On the other way, since
$C_\tau(I_\tau(\theta C_\tau(\mu, \iota) \bigvee C_\tau(I_\tau(\theta C_\tau(\mu, \iota), \iota), \iota), \iota), \iota)$

$\leq C_\tau(I_\tau(\theta C_\tau(\mu,\iota)\vee\theta C_\tau(\theta I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota),\iota),\iota)$
$= C_\tau(I_\tau(\theta C_\tau(\mu,\iota)\vee\theta C_\tau(\theta C_\tau(\theta I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota),\iota),\iota),\iota)$
$= C_\tau(I_\tau(\theta C_\tau(\mu,\iota)\vee\theta C_\tau(\theta I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota),\iota),\iota)$
$= C_\tau(I_\tau(\theta C_\tau(\theta I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota),\iota),\iota)$
$= C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota)$
$\leq \mu\bigvee C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota),$
Then
$\mu\bigvee C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota)$ is ι-fM^*c containing μ.
Hence
$M^*C_\tau(\mu,\ \iota)\leq\mu\bigvee C_\tau(I_\tau(\theta C_\tau(\mu,\ \iota),\ \iota),\ \iota)$.
Thus, we obtain

$$M^*C_\tau(\mu,\ \iota)=\mu\bigvee C_\tau(I_\tau(\theta C_\tau(\mu,\ \iota),\ \iota),\ \iota).$$

□

Theorem 3.11. *Let $(X,\ \tau)$ be a fts. Let $\mu\in I^X$ and $\iota\in I_o$.*

(i) μ is ι-fM^*o iff $\mu=M^*I_\tau(\mu,\ \iota)$.

(ii) μ is ι-fM^*c iff $\mu=M^*C_\tau(\mu,\ \iota)$.

Proof. (i) Let μ be an ι-fM^*o, then
$M^*I_\tau(\mu,\ \iota)=\bigvee\{\nu:\mu\geq\nu,\ \nu \text{ is a } \iota\text{-}fM^*o\}$ = μ.

Conversely, let $\mu=M^*I_\tau(\mu,\ \iota)$, since $M^*I_\tau(\mu,\ \iota)$ is the arbitrary union of ι-fM^*o then μ is ι-fM^*o.

(ii) It is similar to part (i). □

Theorem 3.12. *Let $(X,\ \tau)$ be a fts. For $\mu\in I^X$ and $\iota\in I_0$ we have*

(i) $M^*I_\tau(\overline{1}-\mu,\ \iota)=\overline{1}-M^*C_\tau(\mu,\ \iota)$.

(ii) $M^*C_\tau(\overline{1}-\mu,\ \iota)=\overline{1}-M^*I_\tau(\mu,\ \iota)$.

Proof. By Theorem 3.10 and 3.11, we have for all $\mu\in I^X$ and $\iota\in I_0$, (i) $M^*I_\tau(\overline{1}-\mu,\ \iota)$

$$\begin{aligned}&=(\overline{1}-\mu)\wedge I_\tau(C_\tau(\theta I_\tau(\overline{1}-\mu,\iota),\iota),\iota).\\&=(\overline{1}-\mu)\vee(\overline{1}-C_\tau(I_\tau(\theta C_\tau(\mu,\ \iota),\iota),\iota)).\\&=\overline{1}-(\mu\vee C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota)).\\&=\overline{1}-M^*C_\tau(\mu,\ \iota).\end{aligned}$$

(ii) $M^*C_\tau(\overline{1}-\mu,\ \iota)$

$$\begin{aligned}&=(\overline{1}-\mu)\vee C_\tau(I_\tau(\theta C_\tau(\overline{1}-\mu,\ \iota),\iota),\iota).\\&=(\overline{1}-\mu)\vee(\overline{1}-I_\tau(C_\tau(\theta I_\tau(\mu,\ \iota),\iota),\iota)).\\&=\overline{1}-(\mu\wedge I_\tau(C_\tau(\theta I_\tau(\mu,\ \iota),\iota),\iota)).\\&=\overline{1}-M^*I_\tau(\mu,\ \iota).\end{aligned}$$

□

Theorem 3.13. *Let $(X,\ \tau)$ be a fts. Let $\mu\in I^X$ and $\iota\in I_o$, the following statements hold:*

(i) $M^*C_\tau(\overline{0},\ \iota)=\overline{0}$ and $M^*I_\tau(\overline{1},\ \iota)=\overline{1}$.

(ii) $I_\tau(\mu,\ \iota)\leq M^*I_\tau(\mu,\ \iota)\leq\mu\leq M^*C_\tau(\mu,\ \iota)\leq C_\tau(\mu,\ \iota)$.

(iii) $\mu\leq\nu\Rightarrow M^*I_\tau(\mu,\ \iota)\leq M^*I_\tau(\nu,\ \iota)$ and $M^*C_\tau(\mu,\ \iota)\leq M^*C_\tau(\nu,\ \iota)$.

(iv) $M^*C_\tau(M^*C_\tau(\mu,\ \iota),\ \iota)=M^*C_\tau(\mu,\ \iota)$ and $M^*I_\tau(M^*I_\tau(\mu,\ \iota),\ \iota)=M^*I_\tau(\mu,\ \iota)$.

(v) $M^*C_\tau(\mu,\ \iota)\vee M^*C_\tau(\nu,\ \iota)\leq M^*C_\tau(\mu\vee\nu,\ \iota)$. $M^*I_\tau(\mu,\ \iota)\vee M^*I_\tau(\nu,\ \iota)\leq M^*I_\tau(\mu\vee\nu,\ \iota)$.

(vi) $M^*C_\tau(\mu,\ \iota)\wedge M^*C_\tau(\nu,\ \iota)\geq M^*C_\tau(\mu\wedge\nu,\ \iota)$ and $M^*I_\tau(\mu,\ \iota)\wedge M^*I_\tau(\nu,\ \iota)\geq M^*I_\tau(\mu\wedge\nu,\ \iota)$.

Proof. (i), (ii) and (iii) are trivial from the Definitions of M^*C_τ and M^*I_τ.

(iv) By Theorem 3.10 and 3.11, $M^*C_\tau(M^*C_\tau(\mu,\ \iota),\ \iota)=C_\tau(I_\tau(\theta C_\tau(M^*C_\tau(\mu,\ \iota),\ \iota),\iota),\iota)$

$$\begin{aligned}&=C_\tau(I_\tau(\theta C_\tau(\mu\vee C_\tau(I_\tau(\theta C_\tau(\mu,\ \iota),\ \iota),\ \iota),\ \iota),\ \iota),\ \iota).\\&\leq C_\tau(I_\tau(\theta C_\tau(\mu,\iota)\vee\theta C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota),\iota),\iota).\\&\leq C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota).\\&\leq M^*C_\tau(\mu,\iota).\end{aligned}$$

But
$$M^*C_\tau(\mu,\iota)\leq M^*C_\tau(M^*C_\tau(\mu,\iota),\iota).$$

Hence
$M^*C_\tau(\mu,\iota)=M^*C_\tau(M^*C_\tau(\mu,\iota),\iota)$.
$M^*I_\tau(M^*I_\tau(\mu,\iota),\iota)=I_\tau(C_\tau(\theta I_\tau(M^*I_\tau(\mu,\iota),\iota),\iota),\iota)$

$$\begin{aligned}&=I_\tau(C_\tau(\theta I_\tau(\mu\wedge I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota),\iota),\iota),\iota).\\&\geq I_\tau(C_\tau(\theta I_\tau(\mu,\iota)\wedge\theta I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota),\iota),\iota).\\&\geq I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota)\\&\geq M^*I_\tau(\mu,\ \iota).\end{aligned}$$

But $M^*I_\tau(\mu,\ \iota)\geq M^*I_\tau(M^*I_\tau(\mu,\iota),\iota)$. Hence
$M^*I_\tau(\mu,\iota)=M^*I_\tau(M^*I_\tau(\mu,\iota),\iota)$.

(v) By Theorem 3.10 and 3.11, we have,
$M^*C_\tau(\mu,\iota)\vee M^*C_\tau(\nu,\iota)$
$=[\mu\vee C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota)]\vee[\nu\vee C_\tau(I_\tau(\theta C_\tau(\nu,\iota),\iota),\iota)]$
$=(\mu\vee\nu)\vee(C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota)\vee C_\tau(I_\tau(\theta C_\tau(\nu,\iota),\iota),\iota))$
$\leq(\mu\vee\nu)\vee C_\tau(I_\tau(\theta C_\tau(\mu\vee\nu,\iota),\iota),\iota)$
$\leq M^*C_\tau(\mu\vee\nu,\iota)$.
Hence $M^*C_\tau(\mu,\iota)\vee M^*C_\tau(\nu,\iota)\leq M^*C_\tau(\mu\vee\nu,\iota)$.
$M^*I_\tau(\mu,\ \iota)\vee M^*I_\tau(\nu,\ \iota)$
$=[\mu\vee I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota)]\vee[\nu\vee I_\tau(C_\tau(\theta I_\tau(\nu,\iota),\iota),\iota)]$
$=(\mu\vee\nu)\vee(I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota)\vee I_\tau(C_\tau(\theta I_\tau(\nu,\iota),\iota),\iota))$
$\leq(\mu\vee\nu)\vee I_\tau(C_\tau(\theta I_\tau(\mu\vee\nu,\iota),\iota),\iota)$
$\leq M^*I_\tau(\mu\vee\nu,\ \iota)$.
Hence $M^*I_\tau(\mu,\ \iota)\vee M^*I_\tau(\nu,\ \iota)\leq M^*I_\tau(\mu\vee\nu,\ \iota)$.

(vi) $M^*C_\tau(\mu\wedge\nu,\ \iota)$
$=(\mu\wedge\nu)\wedge C_\tau(I_\tau(\theta C_\tau(\mu\wedge\nu,\iota),\iota),\iota)$
$=(\mu\wedge\nu)\wedge C_\tau(I_\tau(\theta C_\tau(\mu,\iota)\wedge\theta C_\tau(\nu,\iota),\iota),\iota)$
$\leq[\mu\wedge C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota)]\wedge[\nu\wedge C_\tau(I_\tau(\theta C_\tau(\nu,\iota),\iota),\iota)]$
$=M^*C_\tau(\mu,\iota)\wedge M^*C_\tau(\nu,\iota)$.
Hence $M^*C_\tau(\mu,\ \iota)\wedge M^*C_\tau(\nu,\ \iota)\geq M^*C_\tau(\mu\wedge\nu,\ \iota)$.

$$M^*I_\tau(\mu \wedge \nu,\ \iota)$$
$$= (\mu \wedge \nu) \wedge I_\tau(C_\tau(\theta I_\tau(\mu \wedge \nu,\ \iota),\ \iota),\ \iota)$$
$$= (\mu \wedge \nu) \wedge I_\tau(C_\tau(\theta I_\tau(\mu,\iota) \wedge \theta I_\tau(\nu,\iota),\iota),\iota)$$
$$\leq [\mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota)] \wedge [\nu \wedge I_\tau(C_\tau(\theta I_\tau(\nu,\iota),\iota),\iota)]$$
$$= M^*I_\tau(\mu,\ \iota) \wedge M^*I_\tau(\nu,\ \iota).$$

Hence $M^*I_\tau(\mu,\iota) \wedge M^*I_\tau(\nu,\iota) \geq M^*I_\tau(\mu \wedge \nu,\ \iota)$. □

Lemma 3.14. *Let $(X,\ \tau)$ be a sfts. Let $\mu \in I^X$ and $\iota \in I_o$, then*

(i) $M^*C_\tau(\mu,\iota) = \mu \vee \theta sI_\tau(\theta C_\tau(\mu,\iota),\iota)$.

(ii) $M^*I_\tau(\mu,\iota) = \mu \wedge \theta sC_\tau(\theta I_\tau(\mu,\iota),\iota)$.

Proof. (i) From Lemma 2.14(4),
$\mu \vee \theta sI_\tau(\theta C_\tau(\mu,\iota),\iota)$

$$= \mu \vee (C_\tau(I_\tau(\theta C_\tau(\mu,\iota),\iota),\iota))$$
$$= M^*C_\tau(\mu,\iota).$$

(ii) From Lemma 2.14(3),
$\mu \wedge \theta sC_\tau(\theta I_\tau(\mu,\iota),\iota)$

$$= \mu \wedge (I_\tau(C_\tau(\theta I_\tau(\mu,\iota),\iota),\iota))$$
$$= M^*I_\tau(\mu,\iota).$$

□

Theorem 3.15. *The following are equivalent for a subset μ of a sfts $(X,\ \tau)$.*

(i) μ *is an* ι*-fM***o set.*

(ii) $\mu \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$.

(iii) $\theta sC_\tau(\mu,\ \iota) = \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$.

Proof. (i) ⇒ (ii) : Let μ be an ι-fM^*o set.
Then by Theorem 3.13, $\mu = M^*I_\tau(\mu,\ \iota)$.
By Lemma 3.14,

$$\mu = \mu \wedge \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$$
$$\leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota).$$

Hence
$\mu \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$.
(ii) ⇒ (i) : Let $\mu \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$. This implies that $\mu \leq \mu \wedge \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) = M^*I_\tau(\mu,\ \iota)$.
Thus $\mu \leq M^*I_\tau(\mu,\ \iota)$ and hence

$$\mu = M^*I_\tau(\mu,\ \iota),$$

therefore μ is ι-fM^*o.
(ii) ⇒ (iii) : Let $\mu \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$. Then $\theta sC_\tau(\mu,\ \iota) \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$. But $\theta I_\tau(\mu,\ \iota) \leq \mu$.
Hence $\theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) \leq \theta sC_\tau(\mu,\ \iota)$.
Thus $\theta sC_\tau(\mu,\ \iota) = \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$.
(iii) ⇒ (ii) : Let $\theta sC_\tau(\mu,\ \iota) = \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$.
Then $\theta sC_\tau(\mu,\ \iota) \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$.
But $\mu \leq \theta sC_\tau(\mu,\ \iota)$ and therefore

$$\mu \leq \theta sC_\tau(\theta I_\tau(\mu,\ \iota),\ \iota).$$

□

Theorem 3.16. *The following are equivalent for μ of a sfts $(X,\ \tau)$.*

(i) μ *is an* ι*-fM***c set.*

(ii) $\mu \geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$.

(iii) $\theta sI_\tau(\mu,\ \iota) = \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$.

Proof. (i) ⇒ (ii) : Let μ be an ι-fM^*c set. Then by Theorem 3.13, $\mu = M^*C_\tau(\mu,\ \iota)$. By Lemma 3.14,

$$\mu = \mu \vee \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$$
$$\geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota).$$

Hence $\mu \geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$.
(ii) ⇒ (i) : Let $\mu \geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$. This implies that

$$\mu \geq \mu \vee \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$$
$$= M^*C_\tau(\mu,\ \iota).$$

Hence $\mu \geq M^*C_\tau(\mu,\ \iota)$ and thus $\mu = M^*C_\tau(\mu,\ \iota)$ and therefore μ is ι-fM^*c.
(ii) ⇒ (iii) : Let $\mu \geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$.
Then $\theta sI_\tau(\mu,\ \iota) \geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$.
But $\theta C_\tau(\mu,\ \iota) \geq \mu$.
Hence $\theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota) \geq \theta sI_\tau(\mu,\ \iota)$.
Thus

$$\theta sI_\tau(\mu,\ \iota) = \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota).$$

(iii) ⇒ (ii) : Let $\theta sI_\tau(\mu,\ \iota) = \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$. Then $\theta sI_\tau(\mu,\ \iota) \geq \theta sI_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)$. But $\mu \geq \theta sI_\tau(\mu,\ \iota)$ and therefore $\mu \geq \theta sI_\tau(\theta C_\tau(\mu,\ \iota),\ \iota)$. □

Theorem 3.17. *Let $(X,\ \tau)$ be a sfts. For $\mu,\ \nu \in I^X$ and $\iota \in I_0$. then,*

(i) If $\tau(\nu) \geq \iota$ where ν is a crisp subset and μ is an ι-fM*o set, then $\mu \wedge \nu$ is an ι-fM*o set.

(ii) If $\tau(\overline{1} - \nu) \geq \iota$ where ν is a crisp subset and μ is an ι-fM*c set, then $\mu \vee \nu$ is an ι-fM*c set.

Proof. (i) Let μ be ι-fM^*o and $\nu \in I^X$ with $\tau(\nu) \geq \iota$ which is a crisp subset. Then

$$\mu \wedge \nu \leq I_\tau(C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota),\ \iota) \wedge \nu.$$
$$\leq I_\tau(C_\tau(\theta I_\tau(\mu \wedge \nu,\ \iota),\ \iota),\ \iota).$$

Hence $\mu \wedge \nu$ is ι-fM^*o.
(ii) Let μ be ι-fM^*c and $\nu \in I^X$ with $\tau(\overline{1} - \nu) \geq \iota$ which is a crisp subset. Then

$$\mu \vee \nu \geq C_\tau(I_\tau(\theta C_\tau(\mu,\ \iota),\ \iota),\ \iota) \vee \nu.$$
$$\geq C_\tau(I_\tau(\theta C_\tau(\mu \vee \nu,\ \iota),\ \iota),\ \iota).$$

Hence $\mu \vee \nu$ is ι-fM^*c. □

Theorem 3.18. *Let $(X,\ \tau)$ be a sfts. For $\mu,\ \nu \in I^X$ and $\iota \in I_0$.*

(i) μ *is* ι*-f*M^**o iff* $1-\mu$ *is* ι*-f*M^**c.*

(ii) *If* $\tau(\mu) \geq \iota$, *then* μ *is* ι*-f*M^**o set.*

(iii) $I_\tau(\mu,\ \iota)$ *is an* ι*-f*M^**o set.*

(iv) $C_\tau(\mu,\ \iota)$ *is an* ι*-f*M^**c set.*

Proof. (i) and (ii) are trivial.

(iii) From the Definition of I_τ of Theorem 2.5 and Definition 2.2, since $\tau(I_\tau(\mu,\ \iota)) \geq \iota$, by (ii) $I_\tau(\mu,\ \iota)$ is an ι-fM^*o set.

(iv) Since $\overline{1} - C_\tau(\mu,\ \iota) = I_\tau(\overline{1}-\mu,\ \iota)$ from Theorem 2.5, hence we have $\tau(\overline{1} - C_\tau(\mu,\ \iota)) \geq \iota$. Hence by (ii), we have $\overline{1} - C_\tau(\mu,\ \iota)$ is ι-fM^*o. By (i) $C_\tau(\mu,\ \iota)$ is an ι-fM^*c set. □

Theorem 3.19. *Let* $(X,\ \tau)$ *be a sfts,* $\mu,\ \nu \in I^X$ *and* $\iota \in I_0$.

(i) *If* μ *is* ι*-fMo with* $\tau(\overline{1}-\mu) \geq \iota$, *then* μ *is* ι*-f*M^**o*

(ii) *If* μ *is* ι*-fMc with* $\tau(\mu) \geq \iota$, *then* μ *is* ι*-f*M^**o*

(iii) *If* μ *is* ι*-f*θ*so with* $\tau(\mu) \geq \iota$, *then* μ *is* ι*-f*M^**o.*

(iv) *If* μ *is* ι*-f*θ*sc with* $\tau(\overline{1}-\mu) \geq \iota$, *then* μ *is* ι*-f*M^**c.*

Proof. We prove (i) and (iii). Other results have similar proofs.

(i) Let μ be an ι-fMo set and $\tau(\overline{1}-\mu) \geq \iota$.
Then $\delta C_\tau(\mu,\ \iota) = \mu$.

$$\begin{aligned}
\mu &\leq C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) \vee I_\tau(\delta C_\tau(\mu,\ \iota),\ \iota)\\
&= C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota) \vee I_\tau(\mu,\ \iota)\\
&\leq C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)\\
&\leq I_\tau(C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota),\ \iota).
\end{aligned}$$

Hence μ is an ι-fuzzy M^*-open set.

(iii) Let μ be an ι-fθso set and $\tau(\mu) \geq \iota$.

$$\begin{aligned}
\mu &\leq C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota)\\
&\leq I_\tau(C_\tau(\theta I_\tau(\mu,\ \iota),\ \iota),\ \iota)
\end{aligned}$$

Thus μ is an ι-fuzzy M^*-open set. □

4. Conclusion

In this paper, we introduce the idea of ι-fuzzy M^*-open (resp. ι-fuzzy M^*-closed) sets in fuzzy topological spaces in the sense of $\hat{S}$ostak's. Also, we introduce ι-fuzzy M^*-interior (resp. ι-fuzzy M^*-closure) and investigate some of their properties. Moreover, we investigate the relationships between ι-fuzzy open, ι-fuzzy θ-semiopen, ι-fuzzy θ-open, ι-fuzzy δ-semiopen, ι-fuzzy δ-preopen, ι-fuzzy a-open, ι-fuzzy e-open and ι-fuzzy e^*-open in $\hat{S}$ostak's fuzzy topological spaces.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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