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Abelian and representation theorem for generalized Sumudu transform

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Abstract

In this paper, we proved the Abelian theorems(Initial value and final value) and representation theorem for Sumudu transformable generalized functions.

Keywords

Sumudu Transform, Generalized functions, Distribution.

AMS Subject Classification

44A15, 44A20.

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Contents

1. Introduction

In the literature we can see that many researcher have extended various integral transforms to the spaces of generalized functions and studied their operational calculus. Watugala[23] has introduced a new integral transform namely the Sumudu transform and applied it to solve the physical phenomenon in science and engineering. The fundamental properties of Sumudu transform can be seen in [22–26]. Moreover, the applications of Sumudu transform without resorting to a new frequency domain are presented in [1–8, 10–13]. Watugala[25] has extended the Sumudu transform to the functions of two variables with emphasis on solutions to partial differential equations. In[6], we can have the applications to convolution type integral equations along with a Laplace-Sumudu duality. Sadikali[18] presented the abelian theorem for classical Sumudu transform.

The exhaustive literature survey revels that many authors have contributed their work in the extension of Sumudu transform to a generalized function and Bohemians. In [14, 20] we can have generalized Sumudu transform and its fundamental properties. In[9] authors have proved the characterization theorem for generalized Natural transform. The distribution theory provides powerful analytical technique to solve many problems that arises in the applied field. The aim of this paper is to prove abelian theorems and representation theorem for Sumudu transformable generalized functions.

2. Basic Definitions and Preliminary Results

Definition 2.1. ***Sumudu Transformation***
For the function $f(t)$ the Sumudu transform is defined by Watugala[21] as

$$\mathbb{S}[f(t)] = G(u) = \int_0^\infty e^{-t} f(ut)dt \qquad u \in (-\tau_1, \tau_2) \quad (2.1)$$

provided the integral on the right hand side exists. The Sumudu transform of functions $f(t)$ $(t \geq 0)$ are come to exists which are piecewise continuous and of exponential order defined over the set

$\mathfrak{A}$=$[f(t)/\exists\, M,\ \tau_1,\ \tau_2 > 0,\ |f(t)| < M\, e^{\frac{|t|}{\tau_j}}$, *if* $t \in (-1)^j \times [0,\infty)$]

The above transform can be reduced to following form with suitable change in the variable

$$\mathbb{S}[f(t)] = G(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f(t)dt \tag{2.2}$$

The inverse Sumudu transform of function $G(u)$ is denoted by symbol $\mathbb{S}^{-1}[G(u)] = f(t)$ and is defined with Bromwich contour integral[16]

$$\mathbb{S}^{-1}[G(u)] = f(t) = \lim_{T\to\infty}\frac{1}{2\Pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}G(u)du \tag{2.3}$$

Definition 2.2. ***Natural Transformation***
Natural transform of the function $f(t)$ is denoted by symbol $\mathbb{N}[f(t)] = R(s,u)$ where s and u are the transform variables and is defined by an integral equation[15]

$$\mathbb{N}[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)dt \tag{2.4}$$

where $Re(s) > 0, u \in (\tau_1, \tau_2)$, the function $f(t)$ is sectionwise continuous, exponential order and defined over the set $\mathfrak{A}$

If $R(s,u)$ is the Natural transform, $F(s)$ is the Laplace transform and $G(u)$ is Sumudu transform of the function $f(t) \in A$, then the Sumudu-Laplace and Natural-Sumudu duality is given as

$$G(\frac{1}{s}) = sF(s) \qquad F(\frac{1}{u}) = uG(u) \tag{2.5}$$

and

$$\mathbb{N}[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)dt = \frac{1}{s}G(\frac{u}{s}) \tag{2.6}$$

In comparison with the Laplace transform, the Sumudu transform have a units preserving property and hence is used to solve problems without resorting to the frequency domain.

3. Generalized Sumudu Transform

In[14], authors has extended the Sumudu transform to certain spaces of distributions. Here we list some required results
Testing function space $\mathfrak{D}_{a,b}$
Let $\mathfrak{D}_{a,b}$ denotes the space of all complex valued smooth functions $\phi(t)$ on $-\infty < t < \infty$ on which the functions $\gamma_k(\phi)$ defined by

$$\gamma_k(\phi) \triangleq \gamma_{a,b,k}(\phi) \triangleq \underset{0<t<\infty}{Sup.}\, |K_{a,b}(t)D^k(t)| < \infty \tag{3.1}$$

where

$$K_{a,b}(t) = \begin{cases} e^{at} & 0 \le t < \infty \\ e^{bt} & -\infty < t < 0. \end{cases}$$

The space $\mathfrak{D}_{a,b}$ is linear space under the pointwise addition of function and their multiplication by complex numbers. Each γ_k is clearly a seminorm on $\mathfrak{D}_{a,b}$ and γ_0 is a norm. We assign the topology generated by the sequence of seminorm $(\gamma_k)_{k=0}^\infty$ there by making it a countably multinormed space. Note that for each fixed u the kernel $\frac{1}{u}e^{\frac{-t}{u}}$ as a function of t is a member of $\mathfrak{D}_{a,b}$ if and only if $a < Re(\frac{1}{u}) < b$. With the usual argument we can show that $\mathfrak{D}_{a,b}$ is complete and hence a Frechet space. $\mathfrak{D}'_{a,b}$ denotes the dual of $\mathfrak{D}_{a,b}$ i.e. f is member of $\mathfrak{D}'_{a,b}$ if and only if it is continuous linear function on $\mathfrak{D}_{a,b}$. Thus $\mathfrak{D}'_{a,b}$ is a space of generalized functions. Note that the properties of testing function space $\mathfrak{D}_{a,b}$ will follows from[29].
Now we define the generalized Sumudu Transform. Given a generalized Sumudu transformable generalized function $f(t)$, the strip of definition Ω_f for $\mathbb{S}[f]$ is a set in $\mathbb{C}$ defined by $\Omega_f \triangleq \{u : \omega_1 < Re(\frac{1}{u}) < \omega_2\}$ since f or each $u \in \Omega_f$ the kernel $\frac{1}{u}e^{\frac{-t}{u}}$ as a function of t is a member of $\mathfrak{D}'_{\omega_1,\omega_2}$.

For $f \in \mathfrak{D}'_{\omega_1,\omega_2}$, we can define the generalized Sumudu transform of $f(t)$ as conventional function

$$G_f(u) \triangleq \mathbb{S}[f(t)] \triangleq < f(t), \frac{1}{u}e^{\frac{-t}{u}} > \tag{3.2}$$

We call Ω_f the region (or strip) of definition for $\mathbb{S}[f(t)]$ and ω_1 and ω_2 are the abscissas of definition. Note that the properties like linearity and continuity of generalized Sumudu transform will follows from[29].
The boundedness property for the generalized Sumudu transform is given by equation

$$| < f(t), \frac{1}{u}e^{\frac{-t}{u}} > | \le \frac{1}{|u|}M.\underset{0\le k\le r}{Max}\,\underset{t}{Sup.}|K_{a,b}(t)D_t^k e^{\frac{-t}{u}}|$$

Definition 3.1. *Let T be a distribution defined in a neighbourhood of a point, then we say that T has a value C at x_0: $T(x_0) = C$, if the distributional limit $T(x_0 + \lambda x)$ exists in a neighbourhood of zero as λ tends to zero and if it is a constant function C. This concept has been introduced by Lojasiewicz.*

4. Main Results

4.1 Abelian Theorem

In this section, we prove the initial value theorem and final value theorem for the Sumudu transform and then we prove these results for Sumudu transformable functions.

4.1.1 Initial Value Theorem

Theorem 4.1. *Let $f(t) \in \mathfrak{L}_1[0,\infty)$ and if*

i) there exists a real number c such that $\int_0^\infty |f(t)e^{\frac{-t}{c}}|dt < \infty$

ii) there exists a complex number α and a real number $n > -1$ such that $\lim_{t\to 0^+}\frac{f(t)}{t^n} = \alpha$ then $\lim_{u\to\infty}\frac{G(u)}{u^n\Gamma(n+1)} = \alpha$

where Γ is the Euler's gamma function.

Proof. We know that for $n > -1 (n \in \mathbb{R})$ and $u > 0$

$$\frac{1}{u}\int_0^\infty t^n e^{\frac{-t}{u}} dt = \Gamma(n+1).u^n \tag{4.1}$$

Using this result and assuming that $y > 0$ we can write

$$\begin{aligned}|\frac{G(u)}{u^n} - \alpha\Gamma(n+1)| &= \frac{1}{u^n}|G(u) - \alpha\Gamma(n+1)u^n| \\ &= \frac{1}{u^{n+1}}|\int_0^\infty e^{\frac{-t}{u}}(f(t) - \alpha t^n)dt|\end{aligned}$$

$$\leq \frac{1}{u^{n+1}}|\int_0^y e^{\frac{-t}{u}}(f(t)-\alpha t^n)dt| + \frac{1}{u^{n+1}}|\int_y^\infty e^{\frac{-t}{u}}(f(t)-\alpha t^n)dt|$$

$$\leq I_1 + I_2$$

Now

$$\begin{aligned}I_1 &\leq \int_0^y e^{(\frac{-t}{u})}(\frac{t}{u})^n d(\frac{t}{u}) \underset{0<t<y}{Sup.}|(\frac{f(t)}{t^n} - \alpha)| \\ &\leq \Gamma(n+1) \underset{0<t<y}{Sup.}|(\frac{f(t)}{t^n} - \alpha)|\end{aligned}$$

Since from (ii) $\lim\limits_{t\to 0^+} \frac{f(t)}{t^n} = \alpha$, then we can choose y so small, so that $|(\frac{f(t)}{t^n} - \alpha)| < \frac{\varepsilon}{2\Gamma(n+1)}$
thus

$$I_1 < \frac{\varepsilon}{2} \tag{4.2}$$

Now we can choose $\frac{1}{c} > 0$ so that $\int_0^\infty |e^{(\frac{-t}{c})}(\frac{f(t)}{t^n} - \alpha)| < \infty$. Then for $u < c$, we have

$$\begin{aligned}I_2 &= \frac{1}{u^{n+1}}|\int_y^\infty e^{-(\frac{1}{u}-\frac{1}{c})t} e^{(\frac{-t}{c})}(f(t) - \alpha t^n)|dt \\ &\leq K\frac{1}{u^{n+1}} e^{-(\frac{1}{u}-\frac{1}{c})y}\end{aligned}$$

where $K(c,\alpha) = \int_0^\infty e^{(\frac{-t}{c})}|f(t) - \alpha t^n|dt$ is a constant.
Now choose u so large, that

$$I_2 < \frac{\varepsilon}{2} \tag{4.3}$$

Hence for sufficiently large u and from (4.2),(4.3), we have that $\lim\limits_{u\to\infty} \frac{G(u)}{u^n\Gamma(n+1)} = \alpha$
Hence the proof. □

Now we prove the initial value theorem for generalized Sumudu transform.

Theorem 4.2. *If $f \in \mathfrak{D}'_{a,b}$ and $\lim\limits_{t\to 0^+} \frac{f(t)}{t^n} = \alpha$ in the sense of Lojasiewicz then*
$| < f(t) - \alpha t^n, \frac{1}{u}e^{\frac{-t}{u}} > | < \varepsilon$ *as* $u \to \infty$

Proof. We have that[19]

$$f(t) - \alpha t^n = D^p F \tag{4.4}$$

where F is a contineous function in a neighbourhood of zero and $D^p F$ is a finite order derivative of contineous function which is distribution by Zemanian[29].
Now using the boundedness properties of generalized function we have

$$\begin{aligned}< f(t), \frac{1}{u}e^{\frac{-t}{u}} > &\leq \frac{1}{|u|}M.\underset{0\leq k\leq r}{Max}\underset{t}{Sup.}|K_{a,b}(t)D_t^k e^{\frac{-t}{u}}| \\ &\leq \frac{1}{|u|}M.\underset{0\leq k\leq r}{Max}\underset{t}{Sup.}|K_{a,b}(t)(\frac{-1}{u})^k e^{\frac{-t}{u}}| \\ &\leq \frac{1}{(|u|)^{k+1}}M.\underset{0\leq k\leq r}{Max}\underset{t}{Sup.}|K_{a,b}(t)e^{\frac{-t}{u}}| \\ &\leq \varepsilon \quad \text{as} \quad u \to \infty\end{aligned}$$

where M, ε, r are constants. □

4.1.2 Final Value Theorem

Theorem 4.3. *Let $f(t) \in \mathfrak{L}_1[0,\infty)$ and if*

i) there exists a real number "c" such that $\int_0^\infty |f(t)e^{\frac{-t}{c}}|dt < \infty$

ii) there exists a complex number α and a real number $n > -1$ such that
$\lim\limits_{t\to\infty} \frac{f(t)}{t^n} = \alpha$ then $\lim\limits_{u\to 0^+} \frac{G(u)}{u^n\Gamma(n+1)} = \alpha$

Where Γ is the Euler's gamma function.

Proof. Assuming that $u > 0$ and $y > 0$ we can write

$$\begin{aligned}|\frac{G(u)}{u^n} - \alpha\Gamma(n+1)| &= \frac{1}{u^n}|G(u) - \alpha\Gamma(n+1)u^n| \\ &= \frac{1}{u^{n+1}}|\int_0^\infty e^{\frac{-t}{u}}(f(t) - \alpha t^n)dt|\end{aligned}$$

$$\leq \frac{1}{u^{n+1}}|\int_0^y e^{\frac{-t}{u}}(f(t)-\alpha t^n)dt| + \frac{1}{u^{n+1}}|\int_y^\infty e^{\frac{-t}{u}}(f(t)-\alpha t^n)dt|$$

$$\leq I_1 + I_2$$

Now we will evaluate the I_1 and I_2 as follows

$$\begin{aligned}I_1 &\leq \int_0^y e^{(\frac{-t}{u})}(\frac{t}{u})^n d(\frac{t}{u}) \underset{0<t<\infty}{Sup.}|(\frac{f(t)}{t^n} - \alpha)| \\ &\leq \Gamma(n+1) \underset{0<t<\infty}{Sup.}|(\frac{f(t)}{t^n} - \alpha)|\end{aligned}$$

Since $\lim\limits_{t\to\infty} \frac{f(t)}{t^n} = \alpha$ then we can choose y so large, so that $|(\frac{f(t)}{t^n} - \alpha)| < \frac{\varepsilon}{2\Gamma(n+1)}$

$$\text{thus } I_1 < \frac{\varepsilon}{2} \tag{4.5}$$

Now we can choose $\frac{1}{c} > 0$ such that $\int_0^\infty |e^{(\frac{-t}{c})}(\frac{f(t)}{t^n} - \alpha)| < \infty$.Then for $u < c$, we have

$$I_2 = \frac{1}{u^{n+1}} |\int_y^\infty e^{-(\frac{1}{u}-\frac{1}{c})t} e^{(\frac{-t}{c})}(f(t) - \alpha t^n)dt|$$
$$\leq K \frac{1}{u^{n+1}} e^{-(\frac{1}{u}-\frac{1}{c})y}$$

where $K(c,\alpha) = \int_0^\infty e^{(\frac{-t}{c})}|f(t) - \alpha t^n|dt$ is a constant.
Now choose u so small that

$$I_2 < \frac{\varepsilon}{2} \tag{4.6}$$

Hence from equation (4.5), (4.6) we get the required result that $\lim_{u\to 0^+} \frac{G(u)}{u^n \Gamma(n+1)} = \alpha$ □

Now we prove the final value theorem for generalized Sumudu transform.

Theorem 4.4. *Assume that $f \in \mathfrak{D}'_{a,b}$ and $f(t)$ can be decomposed into $f = f_1 + f_2$ where f_1 is ordinary function and $f_2 \in \mathscr{E}'(I)$. Also assume that f_1 satisfies the hypothesis of classical final value theorem. If $G_f(u)$ is the generalized Sumudu transform of $f(t)$ then*
$\lim_{u\to 0^+} \frac{G_f(u)}{u^n} = \Gamma(n+1) \lim_{t\to\infty} \frac{f(t)}{t^n}$
where $\mathscr{E}'(I)$ is the space of distributions having compact support with respect to I.

Proof. Since $f = f_1 + f_2$ then we can have $G_f(u) = G_{f_1}(u) + G_{f_2}(u)$
but $G_{f_2}(u) =< f_2(t), \frac{1}{u} e^{\frac{-t}{u}} >$
As $G_{f_2}(u)$ is smooth function and is of slow growth, we can relate the rate of growth of $G_{f_2}(u)$ to the order of $f_2(t)$. Let $\lambda(t) \in \mathscr{E}(I)$ be identically equal to 1 on a neighborhood support of $f_2(t)$

$$|G_{f_2}(u)| = | < f_2(t), \lambda(t) \frac{1}{u} e^{\frac{-t}{u}} > |$$
$$= C \underset{0\leq t<\infty}{Sup} |D_t^r[\lambda(t) \frac{1}{u} e^{\frac{-t}{u}}]|$$
$$= C \underset{0\leq t<\infty}{Sup} \sum_{v=0}^{r} \binom{r}{v} |D_t^{r-v}\lambda(t)||D_t^v \frac{1}{u} e^{\frac{-t}{u}}|$$
$$= C_1 \underset{0\leq t<\infty}{Sup} |\frac{1}{(u)^{v+1}} e^{\frac{-t}{u}}|$$
$$|\frac{G_{f_2}(u)}{u^n}| \leq C_1 \frac{1}{|u^{n+v+1}|} |e^{\frac{-t}{u}}|$$
$$\lim_{u\to 0^+} |\frac{G_{f_2}(u)}{u^n}| \leq \lim_{u\to 0^+} C_1 \frac{1}{|u^{n+v+1}|} |e^{\frac{-t}{u}}|$$

Substituting $\frac{1}{u} = z$ and $n + v + 1 = k$, we have

$$\lim_{u\to 0^+} |\frac{G_{f_2}(u)}{u^n}| \leq C_1 \lim_{z\to\infty} |(z)^k||e^{-tz}|$$
$$\leq C_1 \lim_{z\to\infty} |\frac{(z)^k}{e^{tz}}|$$

Using the L'Hospital rule we can show that right hand side of above inequality tends to 0 as z tends to ∞

$$\text{Hence} \quad \lim_{u\to 0^+} \frac{G_f(u)}{u^n} = \lim_{u\to 0^+} \frac{G_{f_1}(u)}{u^n} + \lim_{u\to 0^+} \frac{G_{f_2}(u)}{u^n}$$
$$= \lim_{u\to 0^+} \frac{G_{f_1}(u)}{u^n}$$

since f_1 is an ordinary function which satisfies the hypothesis of theorem, the required result follows. □

4.2 Representation Theorem

In this section, we prove that every generalized function $f(t) \in \mathfrak{D}'_{a,b}(I)$ can be represented by a finite sum of derivatives of continuous functions on I. This proof is analogous to the method employed in structure theorems for Schwartz distribution.

Theorem 4.5. *Let $f(t) \in \mathfrak{D}'_{a,b}(I)$ then $f(t)$ can be represented as a finite sum*

$$f(t) = \sum_{i=0}^{n} K_i (\frac{d}{dt})^i [k_{a,b} H_i(t)] \tag{4.7}$$

where $H_i(t)$ are continuous functions on I

Proof. For every $f(t) \in \mathfrak{D}'_{a,b}(I)$, there exists a positive constant C and a non-negative integer r such that for all $\phi \in \mathfrak{D}_{a,b}(I)$

$$| < f(t), \phi(t) > | \leq C \underset{0\leq i\leq r}{Max} (\gamma_{a,b,i}(\phi)) \tag{4.8}$$

where $\mathfrak{D}_{a,b}(I)$ is the space of smooth functions with compact support in I, then we have

$$| < f(t), \phi(t) > | \leq C \underset{0\leq i\leq r}{Max} \underset{t}{Sup} |k_{a,b}(t) D^i \phi(t)| \tag{4.9}$$

Now define $\phi_r(t) = k_{a,b}(t)\phi(t)$, clearly $\phi_r(t) \in \mathfrak{D}_{a,b}(I)$. In this case we have that $\phi \to \phi_r$ is a one-to-one linear map of $\mathfrak{D}_{a,b}(I)$ onto itself. $\phi(t) = k_{a,b}(t)^{-1}\phi_r(t)$
$\frac{d\phi}{dt} = P k_{a,b}(t)^{-1}\phi_r(t) + k_{a,b}(t)^{-1} \frac{d\phi_r}{dt}$
where

$$P = \begin{cases} -a & 0 \leq t < \infty \\ -b & -\infty < t < 0. \end{cases}$$

Let $P_1 = Max.[1,P]$ then
$\frac{d\phi}{dt} = P_1 k_{a,b}(t)^{-1}[\phi_r(t) + \frac{d\phi_r}{dt}]$
continuing in this way, we get

$$(\frac{d\phi}{dt})^i = P_i k_{a,b}(t)^{-1} \sum_{q=0}^{i} (\frac{d}{dt})^q \phi_r(t) \tag{4.10}$$

where $P_i = \underset{k}{Max}[\binom{i}{k} P^{k-i}]$
Using above equation in (4.9), we have

$$| < f(t), \phi(t) > | \leq C' \underset{0\leq i\leq r\, t}{Max} Sup| \sum_{q=0}^{i} (\frac{d}{dt})^q \phi_r(t)|$$

$$\leq C'' \underset{0\leq i\leq r\, t}{Max} Sup|(\frac{d}{dt})^i \phi_r(t)| \tag{4.11}$$

where C', C'' are constants.
Now for every $\psi(t) \in \mathfrak{D}(I)$, we have

$$\underset{t}{Sup}|\psi(t)| \leq \underset{t}{Sup} \int_0^t (\frac{d\psi}{dx})dx \leq \|\frac{d\psi}{dx}\|_{L_1(I)} \tag{4.12}$$

where $L_1(I)$ is the space of equivalence classes of Lebesgue integrable functions on I.
Now the equation(4.11) becomes

$$| < f(t), \phi(t) > | \leq C''' \underset{1\leq i\leq r+1}{Max} \|(\frac{d}{dt})^i \phi_r(t)\|_{L_1(I)} \tag{4.13}$$

where C''' is another constant.
Now consider the linear mapping of $\mathfrak{D}_{a,b}(I)$ into $L_1(I)$ as $\psi(t) \to |(\frac{d}{dt})^i \psi(t)|_{1\leq i\leq r+1}$
Since $\mathfrak{D}_{a,b}(I)$ is linear manifold of $L_1(I)$, equation (4.13) can be read as linear functional. The function $\psi(t)$ is continuous on $\mathfrak{D}_{a,b}(I)$ for the topology induced on it by $L_1(I)$. Therefore by using Hahn-Banach theorem, $f(t)$ can be expanded as continuous linear functional on the whole of $L_1(I)$. But dual of $L_1(I)$ is isomorphic with $L_\infty(I)$, dual space of all equivalence classes of complex valued function on I. Hence for each $f(t) \in L_\infty(I)$ there exists an R such that $|f(t)| \leq R$ almost everywhere.
Hence there exists functions $g_i \in L_\infty(I), 1 \leq i \leq r+1$ such that

$$\begin{aligned} | < f(t), \phi(t) > | &= \sum_{i=1}^{r+1} < g_i, (\frac{d}{dt})^i \phi_r(t) > \\ &= \sum_{i=1}^{r+1} < (-1)^i \frac{d}{dt})^i g_i k_{a,b}(t), \phi(t) > \end{aligned}$$

$$f = \sum_{i=1}^{r+1} (-1)^i (\frac{d}{dt})^i g_i k_{a,b}(t) \tag{4.14}$$

Let for each i we set

$$h_i(t) = (-1)^i \int_0^t g_i(x)dx$$

Since $g_i \in L_\infty(I)$, the function $h_i(t)$ are also continuous on I and

$$\begin{aligned} |h_i(t)| &\leq \int_0^t |g_i(x)dx| \\ &\leq |t| \underset{I}{Max} |g_i| \\ &\leq |t| \|g_i\|_{L_\infty(I)} \end{aligned}$$

Also $g_i = (-1)^i (\frac{d}{dt}) h_i$
Hence

$$f = \sum_{i=2}^{r+2} k_{a,b}(t) (\frac{d}{dt})^i h_i \tag{4.15}$$

Let $r+2 = k$ and using the differentiation formulae

$$v(t)(\frac{d}{dt})^i h_i = \sum_{j=0}^{i} (-1)^i \binom{i}{j} [v^j h_i]^{i-j} \tag{4.16}$$

and

$$(ab)^j = \sum_{j=0}^{i} \binom{j}{q} [a^{j-q} b^q] \tag{4.17}$$

We can write equation (4.15) as in (4.7) where the H_i are the continuous functions of h_i and therefore continuous functions on I. □

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