New notions of nano M-open sets

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Abstract

In this paper we introduce the concepts of new class of sets namely nano M-open sets. Also we discuss various topological properties and interconnections with the existing nano sets.

Keywords

nano M-open, nano δ-preopen, nano θ-semiopen, nano e-open sets.

AMS Subject Classification

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Contents

1. Introduction and Preliminaries

Lellis Thivagar [3] introduced the notion of Nano topology (briefly, $\mathfrak{N}\mathfrak{T}$) by using theory approximations and boundary region of a subset of an universe in terms of an equivalence relation on it and also defined Nano closed (briefly, $\mathfrak{N}c$) sets , Nano-interior (briefly, $\mathfrak{N}int$) and Nano-closure (briefly, $\mathfrak{N}cl$) in a nano topological spaces (briefly, $\mathfrak{N}ts$). Carmel [7] discussed some weak forms of $\mathfrak{N}o$ sets and $\mathfrak{N}\theta$ open (briefly, $\mathfrak{N}\theta o$) sets. The notion of M-open sets in topological spaces were introduced by El-Maghrabi and Al-Juhani [1] in 2011 and studied some of their properties. The class of sets namely, M-open sets are playing more important role in topological spaces, because of their applications in various fields of Mathematics and other real fields. By these motivations, we present the concept of nano M-open sets and study their properties and applications in nano topological space.

Proposition 1.1. [2] If (U, R) is an approximation space and $X, Y \subseteq U$, then

(i) $\mathscr{L}_R(X) \subseteq X \subseteq \mathscr{U}_R(X)$.

(ii) $\mathscr{L}_R(\phi) = \mathscr{U}_R(\phi) = \phi$ and $\mathscr{L}_R(U) = \mathscr{U}_R(U) = U$.

(iii) $\mathscr{U}_R(X \cup Y) = \mathscr{U}_R(X) \cup \mathscr{U}_R(Y)$.

(iv) $\mathscr{U}_R(X \cap Y) \subseteq \mathscr{U}_R(X) \cap \mathscr{U}_R(Y)$.

(v) $\mathscr{L}_R(X \cup Y) \supseteq \mathscr{L}_R(X) \cup \mathscr{L}_R(Y)$.

(vi) $\mathscr{L}_R(X \cap Y) = \mathscr{L}_R(X) \cap \mathscr{L}_R(Y)$.

(vii) $\mathscr{L}_R(X) \subseteq \mathscr{L}_R(Y)$ and $\mathscr{U}_R(X) \subseteq \mathscr{U}_R(Y)$, whenever $X \subseteq Y$.

(viii) $\mathscr{U}_R(X^c) = [\mathscr{L}_R(X)]^c$ and $\mathscr{L}_R(X^c) = [\mathscr{U}_R(X)]^c$.

(ix) $\mathscr{U}_R \mathscr{U}_R(X) = \mathscr{L}_R \mathscr{U}_R(X) = \mathscr{U}_R(X)$.

(x) $\mathscr{L}_R \mathscr{L}_R(X) = \mathscr{U}_R \mathscr{L}_R(X) = \mathscr{L}_R(X)$.

Definition 1.2. [7] Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and let $A \subseteq U$ then the nano θ-interior (resp. nano θ-closure) of A is defined and denoted by $\mathfrak{N}int_\theta(A) = \bigcup\{B : B$ is a $\mathfrak{N}o$ set and $\mathfrak{N}cl(B) \subseteq A\}$ (resp. $\mathfrak{N}cl_\theta(A) = \bigcup\{x \in U : \mathfrak{N}cl(B) \cap A \neq \phi, B$ is a $\mathfrak{N}o$ set and $x \in B\}$).

Definition 1.3. [7] A subset A of X is said to be nano θ-open (resp. nano θ-closed) (briefly, $\mathfrak{N}\theta o$ (resp. $\mathfrak{N}\theta c$)) set if $A = \mathfrak{N}int_\theta(A)$ (resp. A^c is a nano θ-open set).

Definition 1.4. [3, 6] Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and $A \subseteq U$. Then A is said to be nano regular open (briefly, $\mathfrak{N}ro$) if $A = \mathfrak{N}int(\mathfrak{N}cl(A))$.

Definition 1.5. [4] Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and let $A \subseteq U$ then the nano δ-interior (resp. nano δ-closure) of A is defined and denoted by $\mathfrak{N}int_\delta(A) = \bigcup\{B : B$ is a $\mathfrak{N}ro$ set and $B \subseteq A\}$ (resp. $\mathfrak{N}cl_\delta (A) = \bigcup\{x \in U : \mathfrak{N}int(\mathfrak{N}cl(B)) \cap A \neq \phi$, B is a $\mathfrak{N}o$ set and $x \in B\}$).

Definition 1.6. [4] A subset A of X is said to be nano δ-open (resp. nano δ-closed) (briefly, $\mathfrak{N}\delta o$ (resp. $\mathfrak{N}\delta c$)) set if $A = \mathfrak{N}int_\delta(A)$ (resp. A^c is a nano δ-open set).

Definition 1.7. [4, 8]Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and $A \subseteq U$. Then A is said to be a nano δ-pre (resp. nano δ-semi, nano e and nano θ-semi) open set (briefly $\mathfrak{N}\delta\mathscr{P}o$ (resp. $\mathfrak{N}\delta\mathscr{S}o$, $\mathfrak{N}eo$ and $\mathfrak{N}\theta\mathscr{S}o$)) if $A \subseteq \mathfrak{N}int(\mathfrak{N}cl_\delta(A))$ (resp. $A \subseteq \mathfrak{N}cl(\mathfrak{N}int_\delta(A))$, $A \subseteq \mathfrak{N}cl(\mathfrak{N}int_\delta(A)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A))$ and $A \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(A))$).

The complements of the above respective open sets are their respective closed sets.

The family of all $\mathfrak{N}\delta\mathscr{P}o$ (resp. $\mathfrak{N}\delta\mathscr{S}o$, $\mathfrak{N}eo$ and $\mathfrak{N}\theta\mathscr{S}o$) sets is denoted by $\mathfrak{N}\delta\mathscr{P}O(U, \tau_R(X))$, (resp. $\mathfrak{N}\delta\mathscr{S}O(U, \tau_R(X))$, $\mathfrak{N}eO(U, \tau_R(X))$ and $\mathfrak{N}\theta\mathscr{S}O(U, \tau_R(X))$) and the family of all nano δ-pre (resp. nano δ-semi, nano e and nano θ-semi) closed (briefly, $\mathfrak{N}\delta\mathscr{P}c$ (resp. $\mathfrak{N}\delta\mathscr{S}c$, $\mathfrak{N}ec$ and $\mathfrak{N}\theta\mathscr{S}c$)) sets is denoted by $\mathfrak{N}\delta\mathscr{P}C(U, \tau_R(X))$, (resp. $\mathfrak{N}\delta\mathscr{S}C(U, \tau_R(X))$, $\mathfrak{N}eC(U, \tau_R(X))$ and $\mathfrak{N}\theta\mathscr{S}C(U, \tau_R(X))$).

Definition 1.8. [4, 8] Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and let $A \subseteq U$ then the nano δ-pre (resp. nano δ-semi, nano e and nano θ-semi) interior of A is the union of all $\mathfrak{N}\delta\mathscr{P}o$ (resp. $\mathfrak{N}\delta\mathscr{S}o$, $\mathfrak{N}eo$ and $\mathfrak{N}\theta\mathscr{S}o$) sets contained in A and denoted by $\mathfrak{N}\mathscr{P}int_\delta(A)$ (resp. $\mathfrak{N}\mathscr{S}int_\delta(A)$, $\mathfrak{N}eint(A)$ and $\mathfrak{N}\mathscr{S}int_\theta(A)$).

Definition 1.9. Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and let $A \subseteq U$ then the nano δ-pre (resp. nano δ-semi, nano e and nano θ-semi) closure of A is the intersection of all $\mathfrak{N}\delta\mathscr{P}c$ (resp. $\mathfrak{N}\delta\mathscr{S}c$, $\mathfrak{N}ec$ and $\mathfrak{N}\theta\mathscr{S}c$) sets containing A and denoted by $\mathfrak{N}\mathscr{P}cl_\delta(A)$ (resp. $\mathfrak{N}\mathscr{S}cl_\delta(A)$, $\mathfrak{N}ecl(A)$ and $\mathfrak{N}\mathscr{S}cl_\theta(A)$).

Definition 1.10. Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and $A \subseteq U$. Then A is said to be a nano θ-pre (resp. nano θ-semi and nano $\theta\beta$) open set (briefly $\mathfrak{N}\theta\mathscr{P}o$ (resp. $\mathfrak{N}\theta\mathscr{S}o$ and $\mathfrak{N}\theta\beta o$)) if $A \subseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(A))$ (resp. $A \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(A))$ and $A \subseteq \mathfrak{N}cl(\mathfrak{N}int(\mathfrak{N}cl_\theta(A)))$).

Proposition 1.11. [8] Let A be a subset of a $\mathfrak{N}TS$ $(U, \tau_R(X))$. Then

(i) $(\mathfrak{N}cl_\theta(A))^c = \mathfrak{N}int_\theta(A^c)$, $(\mathfrak{N}int_\theta(A))^c = \mathfrak{N}cl_\theta(A^c)$.

(ii) $(\mathfrak{N}\mathscr{S}cl_\theta(A))^c = \mathfrak{N}\mathscr{S}int_\theta(A^c)$, $(\mathfrak{N}\mathscr{S}int_\theta(A))^c = \mathfrak{N}\mathscr{S}cl_\theta(A^c)$.

(iii) $(\mathfrak{N}\mathscr{P}cl_\theta(A))^c = \mathfrak{N}\mathscr{P}int_\theta(A^c)$, $(\mathfrak{N}\mathscr{P}int_\theta(A))^c = \mathfrak{N}\mathscr{P}cl_\theta(A^c)$.

(iv) $(\mathfrak{N}ecl(A))^c = \mathfrak{N}eint(A^c)$, $(\mathfrak{N}eint(A))^c = \mathfrak{N}ecl(A^c)$.

(v) $(\mathfrak{N}\beta cl_\theta(A))^c = \mathfrak{N}\beta int_\theta(A^c)$, $(\mathfrak{N}\beta int_\theta(A))^c = \mathfrak{N}\beta cl_\theta(A^c)$.

(vi) $\mathfrak{N}\mathscr{S}cl_\theta(A) = A \cup \mathfrak{N}int(\mathfrak{N}cl_\theta(A))$.

(vii) $\mathfrak{N}\mathscr{S}int_\theta(A) = A \cap \mathfrak{N}cl(\mathfrak{N}int_\theta(A))$.

Throughout this paper, $(U, \tau_R(X))$ is a $\mathfrak{N}ts$ with respect to X where $X \subseteq U$, R is an equivalence relation on U. Then U/R denotes the family of equivalence classes of U by R. All other undefined notions from [2, 3, 5].

2. Nano *M*-open sets

The purpose of this section is to introduce and investigate some properties of nano *M*-open sets in $\mathfrak{N}ts$. Also obtain the relationship between various nano sets.

Definition 2.1. Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and $A \subseteq U$. Then A is said to be a nano *M*-open (resp. nano *M*-closed) set (briefly $\mathfrak{N}\mathscr{M}o$ (resp. $\mathfrak{N}\mathscr{M}c$)) if $A \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta (A) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A))$ (resp. $A \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A))$).

The family of all $\mathfrak{N}\mathscr{M}o$ (resp. $\mathfrak{N}\mathscr{M}c$) sets are denoted by $\mathfrak{N}\mathscr{M}O(U, \tau_R(X))$, (resp. $\mathfrak{N}\mathscr{M}C(U, \tau_R(X))$).

Definition 2.2. Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and let $A \subseteq U$ then the nano *M*-interior of A is the union of all $\mathfrak{N}\mathscr{M}o$ sets contained in A and denoted by $\mathfrak{N}\mathscr{M}int(A)$.

Definition 2.3. Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$ and let $A \subseteq U$ then the nano *M*-closure of A is the intersection of all $\mathfrak{N}\mathscr{M}c$ sets containing A and denoted by $\mathfrak{N}\mathscr{M}cl(A)$.

Proposition 2.4. Let A be subset in a $\mathfrak{N}ts$, $(U, \tau_R(X))$. Then

(i) $\mathfrak{N}int_\theta(A) \subseteq \mathfrak{N}\mathscr{S}int_\theta(A) \subseteq \mathfrak{N}\mathscr{M}int(A) \subseteq \mathfrak{N}eint(A) \subseteq A$.

(ii) $\mathfrak{N}int_\theta(A) \subseteq \mathfrak{N}int_\delta(A) \subseteq \mathfrak{N}int(A) \subseteq \mathfrak{N}\mathscr{P}int_\delta(A) \subseteq \mathfrak{N}\mathscr{M}int(A) \subseteq A$.

(iii) $\mathfrak{N}int_\delta(A) \subseteq \mathfrak{N}\mathscr{S}int_\delta(A) \subseteq \mathfrak{N}eint(A) \subseteq A$.

Proposition 2.5. Let A be subset in a $\mathfrak{N}ts$, $(U, \tau_R(X))$. Then

(i) $\mathfrak{N}cl_\theta(A) \supseteq \mathfrak{N}\mathscr{S}cl_\theta(A) \supseteq \mathfrak{N}\mathscr{M}cl(A) \supseteq \mathfrak{N}ecl(A) \supseteq A$.

(ii) $\mathfrak{N}cl_\theta(A) \supseteq \mathfrak{N}cl_\delta(A) \supseteq \mathfrak{N}cl(A) \supseteq \mathfrak{N}\mathscr{P}cl_\delta(A) \supseteq \mathfrak{N}\mathscr{M}cl(A) \supseteq A$.

(iii) $\mathfrak{N}cl_\delta(A) \supseteq \mathfrak{N}\mathscr{S}cl_\delta(A) \supseteq \mathfrak{N}ecl(A) \supseteq A$.

Remark 2.6. Let A be a subset of a $\mathfrak{N}TS$ $(U, \tau_R(X))$. Then $(\mathfrak{N}\mathscr{M}cl(A))^c = \mathfrak{N}\mathscr{M}int(A^c)$, $(\mathfrak{N}\mathscr{M}int(A))^c = \mathfrak{N}\mathscr{M}cl(A^c)$.

Proposition 2.7. Let $(U, \tau_R(X))$ be a $\mathfrak{N}ts$. Then,

(i) Every $\mathfrak{N}\theta o$ set is $\mathfrak{N}\delta o$ set.

(ii) Every $\mathfrak{N}\theta\mathscr{S}o$ set is $\mathfrak{N}\mathscr{M}o$ set.

(iii) Every $\mathfrak{N}\delta\mathscr{P}o$ set is $\mathfrak{N}\mathscr{M}o$ set.

(iv) Every $\mathfrak{N}\mathscr{M}o$ set is $\mathfrak{N}eo$ set.

The converse of the above propositions need not to be true. The following examples show it.

Example 2.8. Let $U = \{a,b,c,d\}$ with $U/R = \{\{a\},\{d\}, \{b,c\}\}$ and $X = \{b,d\}$. The $\mathfrak{N}t$ $\tau_R(X) = \{U, \phi, \{d\}, \{b,c\}, \{b,c,d\}\}$

  (i) The nano set $\{d\}$ is $\mathfrak{N}\delta o$ set but not $\mathfrak{N}\theta o$ set.

  (ii) The nano set $\{b\}$ is $\mathfrak{N}\mathscr{M}o$ set but not $\mathfrak{N}\theta So$ set.

  (iii) The nano set $\{a,b,c\}$ is $\mathfrak{N}eo$ set but $\mathfrak{N}\mathscr{M}o$ set.

  (iv) The nano set $\{a,d\}$ is $\mathfrak{N}\mathscr{M}o$ set but not $\mathfrak{N}\delta\mathscr{P}o$ set.

From the above discussions and from the discussions made in [4, 8], the following implications are hold for any set in $\mathfrak{N}ts$.
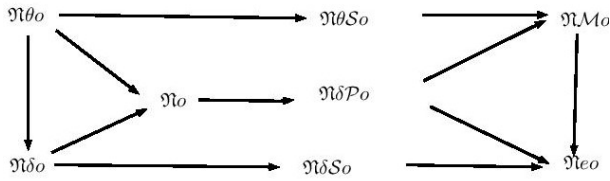
Note: $A \to B$ denotes A implies B, but not conversely.

Proposition 2.9. If A is a $\mathfrak{N}\mathscr{M}o$ set in a $\mathfrak{N}ts$ $(U, \tau_R(X))$ and $\mathfrak{N}int_\theta(A) = \phi$, then A is $\mathfrak{N}\delta\mathscr{P}o$.

Proposition 2.10. Arbitrary union (resp. intersection) of $\mathfrak{N}\mathscr{M}o$ (resp. $\mathfrak{N}\mathscr{M}c$) sets is $\mathfrak{N}\mathscr{M}o$ (resp. $\mathfrak{N}\mathscr{M}c$) set.

    Proof. (1) Let $\{A_i, i \in I\}$ be a family of $\mathfrak{N}\mathscr{M}o$ sets. Then $A_i \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(A_i)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A_i))$, hence $\bigcup_i A_i \subseteq \bigcup_i(\mathfrak{N}cl(\mathfrak{N}int_\theta(A_i)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A_i))) \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(\bigcup_i A_i)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(\bigcup_i A_i))$, for all $i \in I$. Thus $\bigcup_i A_i$ is $\mathfrak{N}\mathscr{M}o$.

    (2) Let $\{A_i, i \in I\}$ be a family of $\mathfrak{N}\mathscr{M}c$ sets. Then $A_i \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(A_i)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A_i))$, hence $\bigcap_i A_i \supseteq \bigcap_i (\mathfrak{N}int(\mathfrak{N}cl_\theta(A_i)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A_i))) \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(\bigcap_i A_i)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(\bigcap_i A_i))$, for all $i \in I$. Thus $\bigcap_i A_i$ is $\mathfrak{N}\mathscr{M}c$.

    The intersection of any two $\mathfrak{N}\mathscr{M}o$ sets need not be a $\mathfrak{N}\mathscr{M}o$ set.

Example 2.11. In Example 2.8, the sets $\{a,b,d\}$ and $\{a,b,c\}$ are $\mathfrak{N}\mathscr{M}o$ sets but their intersection $\{a,b\}$ is not a $\mathfrak{N}\mathscr{M}o$ set.

Lemma 2.12. The following hold for a subset H in a $\mathfrak{N}ts$ $(U, \tau_R(X))$.

  (i) $\mathfrak{N}\mathscr{P}cl_\delta(H) = H \cup \mathfrak{N}cl(\mathfrak{N}int_\delta(H))$ and $\mathfrak{N}\mathscr{P}int_\delta(H) = H \cap \mathfrak{N}int(\mathfrak{N}cl_\delta(H))$,

  (ii) $\mathfrak{N}\mathscr{P}cl_\delta(\mathfrak{N}\mathscr{P}int_\delta(H)) = \mathfrak{N}\mathscr{P}int_\delta(H) \cup \mathfrak{N}cl(\mathfrak{N}int_\delta(H))$ and $\mathfrak{N}\mathscr{P}int_\delta(\mathfrak{N}\mathscr{P}cl_\delta(H)) = \mathfrak{N}\mathscr{P}cl_\delta(H) \cap \mathfrak{N}int(\mathfrak{N}cl_\delta(H))$,

  (iii) $\mathfrak{N}\mathscr{S}int_\delta(H) = H \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(H))$ and $\mathfrak{N}\mathscr{S}cl_\delta(H) = H \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(H))$.

  (iv) $U \setminus (\mathfrak{N}int_\delta(A)) = \mathfrak{N}cl_\delta(U \setminus A)$ and $\mathfrak{N}int_\delta(U \setminus A) = U \setminus \mathfrak{N}cl_\delta(A)$.

Theorem 2.13. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then A is an $\mathfrak{N}\mathscr{M}o$ (resp. $\mathfrak{N}\mathscr{M}c$) set iff $A = \mathfrak{N}\mathscr{S}int_\theta(A) \cup \mathfrak{N}\mathscr{P}int_\delta(A)$ (resp. $A = \mathfrak{N}\mathscr{S}int_\theta(A) \cap \mathfrak{N}\mathscr{P}int_\delta(A)$.)

Proof. Let A be an $\mathfrak{N}\mathscr{M}o$ set. Then $A \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(A)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A))$, hence by Lemma 2.12 $\mathfrak{N}\mathscr{S}int_\theta(A) \cup \mathfrak{N}\mathscr{P}int_\delta(A) = (A \cap \mathfrak{N}cl(\mathfrak{N}int_\theta(A))) \cup (A \cap \mathfrak{N}int(\mathfrak{N}cl_\delta(A))) = A \cap (\mathfrak{N}cl(\mathfrak{N}int_\theta(A))) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A)) = A$.

    Conversely, suppose that $A = \mathfrak{N}\mathscr{S}int_\theta(A) \cup \mathfrak{N}\mathscr{P}int_\delta(A)$. Then by Lemma 2.12 $A = (A \cap \mathfrak{N}cl(\mathfrak{N}int_\theta(A))) \cup (A \cap \mathfrak{N}int(\mathfrak{N}cl_\delta(A))) \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(A) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A)))$. Therefore, A is $\mathfrak{N}\mathscr{M}o$. $\square$

Proposition 2.14. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then A is an $\mathfrak{N}\mathscr{M}c$ set iff $A = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$.

Proof. Let A be an $\mathfrak{N}\mathscr{M}c$ set. Then $A \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A))$, hence by Lemma 2.12 $\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A) = (A \cup \mathfrak{N}int(\mathfrak{N}cl_\theta(A))) \cap (A \cup \mathfrak{N}cl(\mathfrak{N}int_\delta(A))) = A \cup (\mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A))) = A)$.

    Conversely, suppose that $A = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$. Then by Lemma 2.12 $A = (A \cup \mathfrak{N}int(\mathfrak{N}cl_\theta(A))) \cap (A \cup \mathfrak{N}cl(\mathfrak{N}int_\delta(A))) \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(A) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A)))$. Therefore, A is $\mathfrak{N}\mathscr{M}c$. $\square$

Theorem 2.15. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then

  (1) $\mathfrak{N}\mathscr{M}cl(A) = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$,

  (2) $\mathfrak{N}\mathscr{M}int(A) = \mathfrak{N}\mathscr{S}int_\theta(A) \cup \mathfrak{N}\mathscr{P}int_\delta(A)$.

Proof. (1) We claim that $\mathfrak{N}\mathscr{M}cl(A) = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$. Also, $\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A) = (A \cup \mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap A \cup \mathfrak{N}cl(\mathfrak{N}int_\delta(A))) = A \cup (\mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A)))$. But, $\mathfrak{N}\mathscr{M}cl(A)$ is $\mathfrak{N}\mathscr{M}c$, hence $\mathfrak{N}\mathscr{M}cl(A) \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(\mathfrak{N}\mathscr{M}cl(A))) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(\mathfrak{N}\mathscr{M}cl(A))) \supseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A))$. Thus $A \cup (\mathfrak{N}int(\mathfrak{N}cl_\theta(A)) \cap \mathfrak{N}cl(\mathfrak{N}int_\delta(A))) \subseteq A \cup \mathfrak{N}\mathscr{M}cl(A) = \mathfrak{N}\mathscr{M}cl(A)$, therefore, $\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A) \subseteq \mathfrak{N}\mathscr{M}cl(A)$. So, $\mathfrak{N}\mathscr{M}cl(A) = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$.

    (2) We claim that $\mathfrak{N}\mathscr{M}int(A) = \mathfrak{N}\mathscr{S}int_\theta(A) \cup \mathfrak{N}\mathscr{P}int_\delta(A)$. Also, $\mathfrak{N}\mathscr{S}int_\theta(A) \cup \mathfrak{N}\mathscr{P}cl_\delta(A) = (A \cap \mathfrak{N}cl(\mathfrak{N}int_\theta(A)) \cup A \cap \mathfrak{N}int(\mathfrak{N}cl_\delta(A))) = A \cap (\mathfrak{N}cl(\mathfrak{N}int_\theta(A)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A)))$. But, $\mathfrak{N}\mathscr{M}int(A)$ is $\mathfrak{N}\mathscr{M}o$, hence $\mathfrak{N}\mathscr{M}int(A) \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(\mathfrak{N}\mathscr{M}int(A))) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(\mathfrak{N}\mathscr{M}int(A))) \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(A)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A))$. Thus $A \cap (\mathfrak{N}cl(\mathfrak{N}int_\theta(A)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(A))) \supseteq A \cap \mathfrak{N}\mathscr{M}int(A) = \mathfrak{N}\mathscr{M}int(A)$, therefore, $\mathfrak{N}Sint_\theta(A) \cup \mathfrak{N}Pint_\delta(A) \supseteq \mathfrak{N}\mathscr{M}int(A)$. So, $\mathfrak{N}\mathscr{M}int(A) = \mathfrak{N}Sint_\theta(A) \cup \mathfrak{N}Pint_\delta(A)$. $\square$

Theorem 2.16. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then

  (1) A is an $\mathfrak{N}\mathscr{M}o$ set iff $A = \mathfrak{N}\mathscr{M}int(A)$,

  (2) A is an $\mathfrak{N}\mathscr{M}c$ set iff $A = \mathfrak{N}\mathscr{M}cl(A)$.

Proof. (1) Let A be an $\mathfrak{N}\mathscr{M}o$ set. Then by Theorem 2.13, $A = \mathfrak{N}Sint_\theta(A) \cup \mathfrak{N}Pint_\delta(A)$ and by Theorem 2.15, we have $A = \mathfrak{N}\mathscr{M}int(A)$. Conversely, let $A = \mathfrak{N}\mathscr{M}int(A)$. Then by Theorem 2.15, $A = \mathfrak{N}Sint_\theta(A) \cup \mathfrak{N}Pint_\delta(A)$ and by Theorem 2.13, A is $\mathfrak{N}\mathscr{M}o$.

(2) Let A be an $\mathfrak{N}\mathscr{M}c$ set. Then by Theorem 2.13, $A = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$ and by Proposition 2.14, we have $A = \mathfrak{N}\mathscr{M}cl(A)$. Conversely, let $A = \mathfrak{N}\mathscr{M}cl(A)$. Then by Proposition 2.14, $A = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$ and by Theorem 2.13, A is $\mathfrak{N}\mathscr{M}c$. $\qquad\square$

Theorem 2.17. Let A and B be subsets in a $\mathfrak{N}ts$ $(U, \tau_R\ (X))$. Then the following are hold

(1) $\mathfrak{N}\mathscr{M}cl(U\backslash A) = U\backslash\mathfrak{N}\mathscr{M}int(A)$.

(2) $\mathfrak{N}\mathscr{M}int(U\backslash A) = U\backslash\mathfrak{N}\mathscr{M}cl(A)$.

(3) If $A \subseteq B$, then $\mathfrak{N}\mathscr{M}cl(A) \subseteq \mathfrak{N}\mathscr{M}cl(B)$ and $\mathfrak{N}\mathscr{M}int(A) \subseteq \mathfrak{N}\mathscr{M}int(B)$.

(4) $x \in \mathfrak{N}\mathscr{M}cl(A)$ iff there exists an $\mathfrak{N}\mathscr{M}o$ set G and $x \in G$ such that $G \cap A \neq \phi$.

(5) $\mathfrak{N}\mathscr{M}cl(\mathfrak{N}\mathscr{M}cl(A)) = \mathfrak{N}\mathscr{M}cl(A)$ and $\mathfrak{N}\mathscr{M}int(\mathfrak{N}\mathscr{M}int(A)) = \mathfrak{N}\mathscr{M}int(A)$.

(6) $\mathfrak{N}\mathscr{M}cl(A) \cup \mathfrak{N}\mathscr{M}cl(B) \subseteq \mathfrak{N}\mathscr{M}cl(A \cup B)$ and $\mathfrak{N}\mathscr{M}int(A) \cup \mathfrak{N}\mathscr{M}int(B) \subseteq \mathfrak{N}\mathscr{M}int(A \cup B)$.

(7) $\mathfrak{N}\mathscr{M}int(A \cap B) \subseteq \mathfrak{N}\mathscr{M}int(A) \cap \mathfrak{N}\mathscr{M}int(B)$ and $\mathfrak{N}\mathscr{M}cl(A \cap B) \subseteq \mathfrak{N}\mathscr{M}cl(A) \cap \mathfrak{N}\mathscr{M}cl(B)$.

Proof. We prove only (1)-(3) and (5), as others are obvious.

(1) Since $(U\backslash A) \subseteq U$, by Theorem 2.16 $\mathfrak{N}\mathscr{M}cl(U\backslash A) = \mathfrak{N}\mathscr{S}cl_\theta(U\backslash A) \cap \mathfrak{N}\mathscr{P}cl_\delta(U\backslash A)$ and by Lemma 2.12, $\mathfrak{N}\mathscr{M}cl(U\backslash A) = (U\backslash\mathfrak{N}Sint_\theta(A)) \cap (U\backslash\mathfrak{N}Pint_\delta(A)) = U\backslash(\mathfrak{N}S\ int_\theta(A) \cup \mathfrak{N}Pint_\delta(A))$, hence by Theorem 2.16, $\mathfrak{N}\mathscr{M}cl(U\backslash A) = U\backslash\mathfrak{N}\mathscr{M}int(A)$.

(2) Since $U \subseteq (U\backslash A)$, by Theorem 2.16 $\mathfrak{N}\mathscr{M}int(U\backslash A) = \mathfrak{N}Sint_\theta(U\backslash A) \cup \mathfrak{N}Pint_\delta(U\backslash A)$ and by Lemma 2.12, $\mathfrak{N}\mathscr{M}int(U\backslash A) = (U\backslash\mathfrak{N}\mathscr{S}cl_\theta(A)) \cup (U\backslash\mathfrak{N}\mathscr{P}cl_\delta(A)) = U\backslash(\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A))$, hence by Theorem 2.16, $\mathfrak{N}\mathscr{M}\ int(U\backslash A) = U\backslash\mathfrak{N}\mathscr{M}cl(A)$.

(3) Since, $\mathfrak{N}\mathscr{M}cl(A) = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)$ and $A \subseteq B$, $\mathfrak{N}\mathscr{M}cl(A) = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A) \subseteq \mathfrak{N}\mathscr{S}cl_\theta(B) \cap \mathfrak{N}\mathscr{P}cl_\delta(B) = \mathfrak{N}\mathscr{M}cl(B)$.

(5) Since, $\mathfrak{N}\mathscr{M}cl(\mathfrak{N}\mathscr{M}cl(A)) = \mathfrak{N}\mathscr{S}cl_\theta(\mathfrak{N}\mathscr{M}cl(A)) \cap \mathfrak{N}\mathscr{P}cl_\delta(\mathfrak{N}\mathscr{M}cl(A))$, by Theorem 2.16 $\mathfrak{N}\mathscr{S}cl_\theta(\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)) \cap \mathfrak{N}\mathscr{P}cl_\delta(\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)) \subseteq (\mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{S}cl_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))) \cap (\mathfrak{N}\mathscr{P}cl_\delta(\mathfrak{N}\mathscr{S}cl_\theta(A)) \cap \mathfrak{N}\mathscr{P}cl_\delta(A)) = \mathfrak{N}\mathscr{S}cl_\theta(A) \cap \mathfrak{N}\mathscr{P}cl_\delta(A) = \mathfrak{N}\mathscr{M}cl(A)$, hence $\mathfrak{N}\mathscr{M}cl(\mathfrak{N}\mathscr{M}cl(A)) \subseteq \mathfrak{N}\mathscr{M}cl(A)$. But, $\mathfrak{N}\mathscr{M}cl(A) \subseteq \mathfrak{N}\mathscr{M}cl(\mathfrak{N}\mathscr{M}cl(A))$. Therefore, $\mathfrak{N}\mathscr{M}cl(\mathfrak{N}\mathscr{M}cl(A)) = \mathfrak{N}\mathscr{M}cl(A)$. $\qquad\square$

Remark 2.18. The inclusion relation in part (6),(7) of the above theorem cannot be replaced by equality as shown by the following example.

Example 2.19. In Example 2.8,

(i) If $A = \{b\}$, $B = \{d\}$ and $A \cup B = \{b,d\}$, then $\mathfrak{N}\mathscr{M}cl(A) = A$, $\mathfrak{N}\mathscr{M}cl(B) = B$ and $\mathfrak{N}\mathscr{M}cl(A \cup B) = \{a,b,d\}$. Hence, $\mathfrak{N}\mathscr{M}cl(A \cup B) \not\subseteq \mathfrak{N}\mathscr{M}cl(A) \cup \mathfrak{N}\mathscr{M}cl(B)$.

(ii) If $A = \{b,d\}$, $C = \{a,d\}$ and $A \cap B = \{d\}$, then $\mathfrak{N}\mathscr{M}cl(A) = \{a,b,d\}$, $\mathfrak{N}\mathscr{M}cl(C) = C$ and $\mathfrak{N}\mathscr{M}cl(A \cap C) = \{d\}$. Therefore, $\mathfrak{N}\mathscr{M}cl(A) \cap \mathfrak{N}\mathscr{M}cl(c) \not\subseteq \mathfrak{N}\mathscr{M}\ cl(A \cap C)$.

(iii) If $D = \{a\}$, $C = \{d\}$ and $D \cup C = \{a,d\}$, then $\mathfrak{N}\mathscr{M}int(D) = \phi$, $\mathfrak{N}\mathscr{M}int(C) = \{d\}$ and $\mathfrak{N}\mathscr{M}int(D \cup C) = \{a,d\}$. So, $\mathfrak{N}\mathscr{M}int(D \cup C) \not\subseteq \mathfrak{N}\mathscr{M}int(D) \cup \mathfrak{N}\mathscr{M}int(C)$.

Lemma 2.20. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then

(1) $\mathfrak{N}\mathscr{P}int(\mathfrak{N}\mathscr{P}cl_\delta(A)) = \mathfrak{N}\mathscr{P}cl_\delta(A) \cap \mathfrak{N}int(\mathfrak{N}cl(A))$ and $\mathfrak{N}\mathscr{P}cl(\mathscr{P}int_\delta(A)) = \mathscr{P}int_\delta(A) \cup \mathfrak{N}cl(\mathfrak{N}int(A))$.

(2) $\mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A)) = \mathfrak{N}\mathscr{P}cl_\delta(A) \cap \mathfrak{N}int(\mathfrak{N}cl_\theta(A))$ and $\mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A)) = \mathfrak{N}\mathscr{P}int_\delta(A) \cup \mathfrak{N}cl(\mathfrak{N}int_\theta(A))$.

(3) $\mathfrak{N}\mathscr{S}cl_\theta(\mathfrak{N}int_\theta(A)) = \mathfrak{N}\mathscr{S}cl(\mathfrak{N}int_\theta(A)) = \mathfrak{N}int(\mathfrak{N}cl(\mathfrak{N}int_\theta(A)))$

Proposition 2.21. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then:

(1) $\mathfrak{N}\mathscr{M}cl(A) = A \cup \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$.

(2) $\mathfrak{N}\mathscr{M}int(A) = A \cap \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$.

Proof. (1) By Lemma 2.20, $A \cup \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A)) = A \cup (\mathfrak{N}\mathscr{P}cl_\delta(A) \cap \mathfrak{N}int(\mathfrak{N}cl_\theta(A))) = (A \cup \mathfrak{N}\mathscr{P}cl_\delta(A)) \cap (A \cup \mathfrak{N}int(\mathfrak{N}cl_\theta(A))) = \mathfrak{N}\mathscr{P}cl_\delta(A) \cap \mathfrak{N}\mathscr{S}cl_\theta(A) = \mathfrak{N}\mathscr{M}cl(A)$.

(2) By Lemma 2.20, $A \cap \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A)) = A \cap (\mathfrak{N}\mathscr{P}int_\delta(A) \cup \mathfrak{N}cl(\mathfrak{N}int_\theta(A))) = (A \cap \mathfrak{N}\mathscr{P}int_\delta(A)) \cup (A \cap \mathfrak{N}cl(\mathfrak{N}int_\theta(A))) = \mathfrak{N}\mathscr{P}int_\delta(A) \cup \mathfrak{N}\mathscr{S}int_\theta(A) = \mathfrak{N}\mathscr{M}int(A)$. $\qquad\square$

Theorem 2.22. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then A is an $\mathfrak{N}\mathscr{M}o$ set iff $A \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$.

Proof. Let A be an $\mathfrak{N}\mathscr{M}o$ set. Then by Theorem 2.16, $A = \mathfrak{N}\mathscr{M}int(A)$ and by Proposition 2.21, $A = A \cap \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$ and hence, $A \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$.

Conversely, let $A \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$. Then by Proposition 2.21, $A \subseteq A \cap \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A)) = \mathfrak{N}\mathscr{M}int(A)$. so, $A \subseteq \mathfrak{N}\mathscr{M}int(A)$. Then $A = \mathfrak{N}\mathscr{M}int(A)$ and hence, A is $\mathfrak{N}\mathscr{M}o$. $\qquad\square$

Theorem 2.23. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then $A \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$ iff $\mathfrak{N}\mathscr{P}cl_\theta(A) = \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$.

Proof. Let $A \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$. Then $\mathfrak{N}\mathscr{P}cl_\theta(A) \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$ and hence, $\mathfrak{N}\mathscr{P}cl_\theta(A) = \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$.

Conversely, let $\mathfrak{N}\mathscr{P}cl_\theta(A) = \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$. Then $\mathfrak{N}\mathscr{P}cl_\theta(A) \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$ and hence, $A \subseteq \mathfrak{N}\mathscr{P}cl_\theta(\mathfrak{N}\mathscr{P}int_\delta(A))$. $\qquad\square$

Theorem 2.24. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then A is an $\mathfrak{N}\mathscr{M}c$ set iff $\mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A)) \subseteq A$.

Proof. Let A be an $\mathfrak{N}\mathcal{M}c$ set. Then by Theorem 2.16, $A = \mathfrak{N}\mathcal{M}cl(A)$ and by Proposition 2.21, $A = A \cup \mathfrak{N}\mathscr{P}int_\theta (\mathfrak{N}\mathscr{P}cl_\delta(A))$ and hence, $A \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$.

Conversely, let $A \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$. Then by Proposition 2.21, $A \supseteq A \cup \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A)) = \mathfrak{N}\mathcal{M}cl$ (A). so, $A \supseteq \mathfrak{N}\mathcal{M}cl(A)$. Then $A = \mathfrak{N}\mathcal{M}cl(A)$ and hence, A is $\mathfrak{N}\mathcal{M}c$. $\square$

Theorem 2.25. Let A be subset in a $\mathfrak{N}ts$ $(U, \tau_R(X))$. Then $\mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A)) \subseteq A$ iff $\mathfrak{N}\mathscr{P}int_\theta(A) = \mathfrak{N}\mathscr{P}int_\theta$ $(\mathfrak{N}\mathscr{P}cl_\delta(A))$

Proof. $(1) \Rightarrow (2)$. Let A be an $\mathfrak{N}\mathcal{M}c$ set. Then by Theorem 2.16, $A = \mathfrak{N}\mathcal{M}cl(A)$ and by Proposition 2.21, $A = A \cup \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$ and hence, $A \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$.

$(2) \Rightarrow (1)$. Let $A \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$. Then by Proposition 2.21,
$A \supseteq A \cup \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A)) = \mathfrak{N}\mathcal{M}cl(A)$.
so, $A \supseteq \mathfrak{N}\mathcal{M}cl(A)$. Then $A = \mathfrak{N}\mathcal{M}cl(A)$ and hence, A is $\mathfrak{N}\mathcal{M}c$.

$(2) \Rightarrow (3)$. Let $A \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$. Then $\mathfrak{N}\mathscr{P}int_\theta(A) \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$ and hence, $\mathfrak{N}\mathscr{P}int_\theta(A) = \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$.

$(3) \Rightarrow (2)$. Let $\mathfrak{N}\mathscr{P}int_\theta(A) = \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$. Then $\mathfrak{N}\mathscr{P}int_\theta(A) \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$ and hence,$A \supseteq \mathfrak{N}\mathscr{P}int_\theta(\mathfrak{N}\mathscr{P}cl_\delta(A))$. $\square$

Definition 2.26. A subset A of a $\mathfrak{N}ts$, $(U, \tau_R(X))$ is said to be locally $\mathfrak{N}\mathcal{M}c^{**}$ (briefly, $\mathfrak{N}lMc^{**}$) if $A = G \cap F$ for each G is $\mathfrak{N}o$ and F is $\mathfrak{N}\mathcal{M}c$.

Theorem 2.27. Let H be a subset of a $\mathfrak{N}ts$, $(U, \tau_R(X))$. Then H is $\mathfrak{N}lMc^{**}$ iff $H = G \cap \mathfrak{N}\mathcal{M}cl(H)$.

Proof. Since H is a $\mathfrak{N}lMc^{**}$ set, $H = G \cap F$, for each G is $\mathfrak{N}o$ and F is $\mathfrak{N}\mathcal{M}c$, hence $H \subseteq \mathfrak{N}\mathcal{M}cl(H) \subseteq \mathfrak{N}\mathcal{M}cl(F) = F$, thus $H \subseteq G \cap \mathfrak{N}\mathcal{M}cl(H) \subseteq G \cap \mathfrak{N}\mathcal{M}cl(F) = H$. Therefore $H = G \cap \mathfrak{N}\mathcal{M}cl(H)$. Conversely, since $\mathfrak{N}\mathcal{M}cl(H)$ is $\mathfrak{N}\mathcal{M}c$ and $H = G \cap \mathfrak{N}\mathcal{M}cl(H)$, then H is $\mathfrak{N}lMc^{**}$. $\square$

Theorem 2.28. Let A be a $\mathfrak{N}lMc^{**}$ subset of a $\mathfrak{N}ts$, $(U, \tau_R(X))$. Then

(1) $\mathfrak{N}\mathcal{M}cl(A) \backslash A$ is an $\mathfrak{N}\mathcal{M}c$ set.

(2) $(A \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A)))$ is an $\mathfrak{N}\mathcal{M}o$.

(3) $A \subseteq \mathfrak{N}\mathcal{M}\mathfrak{N}int(A \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A)))$.

Proof. (1) If A is $\mathfrak{N}lMc^{**}$ set, then there exist an $\mathfrak{N}o$ set G such that $A = G \cap \mathfrak{N}\mathcal{M}cl(A)$. Hence, $\mathfrak{N}\mathcal{M}cl(A) \backslash A = \mathfrak{N}\mathcal{M}cl(A) \backslash (G \cup \mathfrak{N}\mathcal{M}cl(A))) = \mathfrak{N}\mathcal{M}cl(A) \cap [U \backslash (G \cap \mathfrak{N}\mathcal{M}cl(A))] = \mathfrak{N}\mathcal{M}cl(A) \cap [(U \backslash G) \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A))] = \mathfrak{N}\mathcal{M}cl(A) \cap (U \backslash G)$ which is $\mathfrak{N}\mathcal{M}c$.

(2) From (1), $\mathfrak{N}\mathcal{M}cl(A) \backslash A$ is $\mathfrak{N}\mathcal{M}c$, then $U \backslash [\mathfrak{N}\mathcal{M}cl(A) \backslash A]$ is an $\mathfrak{N}\mathcal{M}o$ set and $U \backslash [\mathfrak{N}\mathcal{M}cl(A) \backslash A] = U \backslash \mathfrak{N}\mathcal{M}cl(A) \cup (U \cap A) = A \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A))$, hence $A \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A))$ is $\mathfrak{N}\mathcal{M}o$.

(3) It is clear that, $A \subseteq (A \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A))) = \mathfrak{N}\mathcal{M}int(A \cup (U \backslash \mathfrak{N}\mathcal{M}cl(A)))$. $\square$

Remark 2.29. The family $\mathfrak{N}\mathcal{M}O(U, \tau_R(X))$ is a supra topology on U.

Proposition 2.30. $\mathfrak{N}\theta\mathscr{S}O(U, \tau_R(X)) \cup \mathfrak{N}\delta\mathscr{P}O(U, \tau_R(X)) \subseteq \mathfrak{N}\mathcal{M}O(U, \tau_R(X))$.

Conclusion

In this paper, we have studied the class of $\mathfrak{N}\mathcal{M}o$ sets and their properties in $\mathfrak{N}ts$.

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