Malaya Journal of Matematik, Vol. S, No. 1, 66-71, 2020
https://doi.org/10.26637/MJM0S20/0013

Degree based partition of the power graph of the finite Abelian group

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Abstract
The power graph of the group G is the graph, whose vertex set is the group G itself, and there is an edge between any two distinct vertices if one is a power of the other. It is denoted as ζ_G. In this paper, we mainly focused only on the undirected power graph of a finite abelian group of an order pq, where p and q are two distinct primes, $p<q$ and $p \neq q$. First, we shall give the generalization result for the vertex partition set and edge partition set based on the degrees of the power graph ζ_G.The generators of the group Z_{pq} can be found from the partition. Then we can derive some types of topological indices of $\zeta_{\overline{G}}$ of the group Z_{pq}.

Keywords
Vertex partition, edge partition, generators of the group, zagreb indices.

AMS Subject Classification
94C15, 05CXX, 20D06, 20F05, 20D05.

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Article History: Received **01** January **2020**; Accepted **12** February **2020**

Contents

1. Introduction

Groups as graphs contains the most merging combination which is used regularly in the algebraic graph theory. The undirected power graph G [1] in which two distinct vertices x and y are adjacent if one is a power of the other. The concept of the power graphs in group is the recent development area of research. It is easy to see that the power graph of a given group is connected. The power graph G of the group G is complete iff G is cyclic group of prime power order. The two abelian group with two isomorphic power graphs are isomorphic [2, 3].

Next, we proceed to generalize the power graph of the finite abelian group of an order pq, $|G| = pq$ where p and q are two distinct prime. A cyclic group is the group that is generated by a single element g .i.e, there exist an element g, such that every element of the group can be written as a power of g. This element g is the generator of the group.

In general case, finding generator of cyclic group is the difficult task. we belive that there is no fast algorithm to find the generator for the larger number of the group. For example, Every element of a cyclic group of the prime order except the identity does generate the whole group. The generators of the cyclic group depends upon the order of the group. By using that, we may consider the power graph of an order may be pq. It is followed that we may find the generators of the large group orders, This method may be suitable to explicitly evaluate all groups under the product of two distinct primes, $p<q$ and $p \neq q$.

The main intuition of this paper is to find the vertex partition and edge partition [8] based on the degree of the vertices of the power graph ζ_G of the group G. we may simplify the methodology for finding the generators of the group G using the vertex partition of the power graph ζ_G .

The power graph may represent the molecular structure of a certain chemical compound and it is mainly associated with the different molecular-biology, specially in the graph neural networks, network navigation, designing network tools etc.,

It is also used to maintain state, the information to capture the adjacency properties of nodes. The molecular descriptors are based on the sense of ultimate outcome of logic and the numerical procedure which converts the preset chemical information into a symbolic representation of molecule will reach a practical number.
Therefore, the topological indices [7] are very essential tool in the field of nanotechnology, drug discovery, computer networks, designing tool, information technology etc., It is mainly motivated to explore classification purpose, such as Quantitative Structure Activity Relationship (QSAR) and Quantitative Structure Property Relation (QSPR) and Fuzzy Lattice Neural Networks (FLNN) [9] etc. since we can derive some expression about the first zagreb indices, second zagreb indices, the first zagreb co indices, second zagreb co indices and also complements of the power graph ζ_G.

2. Preliminaries

Some of the well-known basic definitions which are defined earlier for any simple graph as follows

Definition 2.1. *A partition of a set is a grouping of its elements into non-empty, in such a way that element is included in exactly one subset.*

Definition 2.2. *The First Zagreb Index [6] of a graph ζ_G, denoted by $M_1(\zeta_G)$ is defined as*

$$M_1(\zeta_G) = \sum_{u,v \in E(\zeta_G)} [d(u) + d(v)]$$

Definition 2.3. *The Second Zagreb Index [6] of a graph ζ_G, denoted by $M_2(\zeta_G)$ is defined as*

$$M_2(\zeta_G) = \sum_{u,v \in E(\zeta_G)} [d(u)d(v)]$$

Definition 2.4. *The Third Zagreb Index [4] of a graph ζ_G, denoted by $M_3(\zeta_G)$ is defined as*

$$M_3(\zeta_G) = \sum_{u,v \in E(\zeta_G)} |d(u) - d(v)|$$

Definition 2.5. *The First Zagreb Co-Index [5] of a graph ζ_G, denoted by $\overline{M_1}(\zeta_G)$ is defined as*

$$\overline{M_1}(\zeta_G) = \sum_{u,v \notin E(\zeta_G)} [d(u) + d(v)]$$

Definition 2.6. *The Second Zagreb Co-Index [5] of a graph ζ_G, denoted by $\overline{M_2}(\zeta_G)$ is defined as*

$$\overline{M_2}(\zeta_G) = \sum_{u,v \notin E(\zeta_G)} [d(u)d(v)]$$

The complement of the graph ζ_G has the same vertex set as that of ζ_G if and if they are not adjacent in ζ_G. The relationship between the zagreb indices and co-indices of the graph can be shown in [5] .The set of vertices and set of edges are always denoted by V and E respectively.

Theorem 2.7. *Let ζ be any graph with n vertices and m edges. Then*

1. $M_1(\overline{\zeta}) = M_1(\zeta) + n(n-1)^2 - 4m(n-1)$
2. $\overline{M_1}(\zeta) = 2m(n-1) - M_1(\zeta)$
3. $\overline{M_1}(\overline{\zeta}) = 2m(n-1) - M_1(\zeta)$
4. $M_2\overline{\zeta} = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{2n-3}{2}M_1(\zeta) - M_2(\zeta)$
5. $\overline{M_2}(\zeta) = 2m^2 - \frac{1}{2}M_1(\zeta) + M_2(\zeta)$
6. $\overline{M_2}(\overline{\zeta}) = m(n-1)^2 - (n-1)M_1(\zeta) + M_2(\zeta)$

The above definitions and theorems are the basic informations which are useful in subsequent sections. The computation of different types of zagreb indices of various graph families may be referred in [10, 11].

3. Main Results

Let G be a finite abelian group. The graph ζ_G(V,E) is the power graph of G where V is the vertex set and E is the edge set of the graph ζ_G. We denote the minimum degree as δ = Min $\{d(v)|v\varepsilon V(\zeta_G)\}$ and the maximum degree as Δ = Max $\{d(v)|v\varepsilon V(\zeta_G)\}$. The vertex set and edge set can be partitioned based on the degree as follows [8].
There are three different degrees of the vertices of the power graph ζ_G. According to the degrees, the vertex set V can be partitioned into V_x, V_y, V_z. Assume that $x < y < z$.

$$V_x = \{v \in V(\zeta_G) | x = d(v) = q(p-1)\}$$
$$V_y = \{v \in V(\zeta_G) | y = d(v) = p(q-1)\}$$
$$V_z = \{v \in V(\zeta_G) | z = d(v) = pq - 1\}$$

Now we have

$$V = V_x \cup V_y \cup V_z$$

The cardinality of the vertex partition can be calculated and generalized as follows,

$$|V_x| = p - 1$$
$$|V_y| = q - 1$$
$$|V_z| = pq - p - q + 2$$

$\triangle(\zeta_G) = pq-1$, from the definition of V_x, V_y, V_z
The graph ζ_G of the edge set can be divided into five partitions as follows :

$$E_1 = \{(u,v) \in E | d(u) = d(v) = q(p-1)\}$$
$$E_2 = \{(u,v) \in E | d(u) = d(v) = p(q-1)\}$$
$$E_3 = \{(u,v) \in E | d(u) = p(q-1), d(v) = pq-1\}$$
$$E_4 = \{(u,v) \in E | d(u) = q(p-1), d(v) = pq-1\}$$
$$E_5 = \{(u,v) \in E | d(u) = d(v) = pq-1\}$$

Now we have

$$E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

The cardinality of the edge partition can be calculated and generalized as follows,

$$|E_1| = \frac{(p-1)(p-2)}{2}$$
$$|E_2| = \frac{(q-1)(q-2)}{2}$$
$$|E_3| = \frac{(q-1)(pq-p-q+2)}{2}$$
$$|E_4| = (p-1)(pq-p-q+2)$$
$$|E_5| = (pq-p-q+1)(pq-p-q+2)$$

Using the cardinality of vertex partition and edge partition, we have

$$|V| = pq, |E| = \frac{p^2q^2 - 3pq + 2p + 2q - 2}{2}$$

Remark 3.1. *The generators of the cyclic group Z_{pq} are the non-identity elements of the partition V_z whose cardinality is maximum.*

3.1 Example

Consider the cyclic group Z_{15}, where $p = 3, q = 5$. The power graph of Z_{15} is shown in Figure.1.
The generators of the group will generate the entire set of the group $(Z_{15}, +_{15})$., These are the elements of Z_{15} which are relatively prime to 15.
Therefore, the generator of the group Z_{15}, denoted by N, is given by

$$N = \{1,2,4,7,8,11,13,14\}$$

By the definition of the power graph, the vertices of the power graph ζ_G are the elements of the power graph

$$Z_{15} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$$

.
The degrees of the elements of Z_{15} are

$$= \{14,14,14,12,14,10,12,14,14,12,10,14,12,14,14\}$$

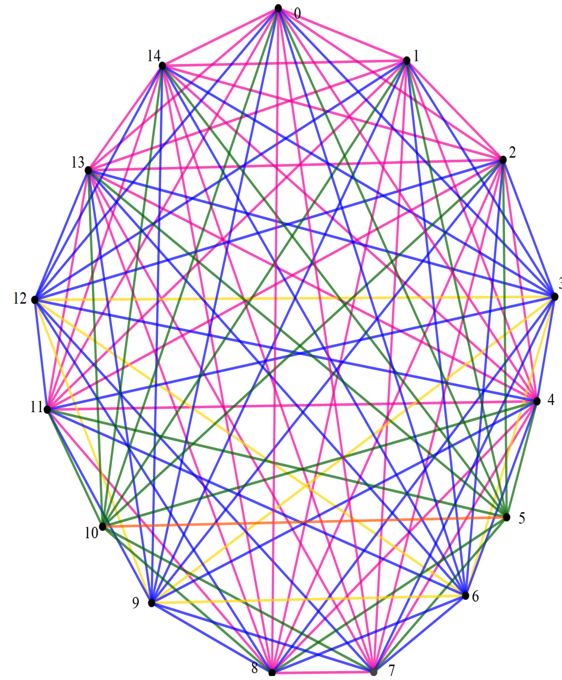

Figure 1. The power graph of Z_{15}

Based on the degrees 10,12,14, the vertex set can be partitioned into V_x, V_y, V_z, where $x = 10, y = 12, z = 14$. Now we have

$$V_x = \{5,10\}$$
$$V_y = \{3,6,9,12\}$$
$$V_z = \{0,1,2,4,7,8,11,13,14\}$$

which is shown in Figure.2.

The cardinality V_z is maximum and hence the nonzero elements of V_z will give us generators of Z_{15} i.e.,

$$N = V_Z \setminus \{0\} = \{1,2,4,7,8,11,13,14\}$$

.

3.2 The zagreb indices of the power graph of the finite abelian Group

In this section, we determine some types of Zagreb Indices of the power graph of the finite abelian group G as follows,

Theorem 3.2. *Let ζ_G be the power graph of the finite abelian group of an order pq. Then its Zagreb indices are given by*

1. $M_1(\zeta_G) = |G|^3 - 6|G|^2 - |G| - (p+q)[(p+q) - 4|G|] + 2$

2. $M_2(\zeta_G) = \frac{1}{2}[|G|^4 - 9|G|^3 + 11|G|^2(p+q) - |G|$. $(13(p+q)+10) + (p+q)[5(p+q)-7] + 2$

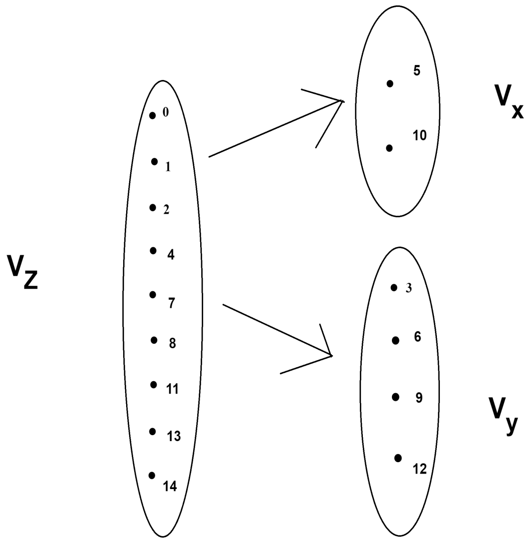

Figure 2. Vertex partition of $\zeta_{15}, V_x \sim V_y \Rightarrow$ Non-adjacency

3. $M_3(\zeta_G) = |G|^2 - 2|G|(p+q+1) + 5|G| + 3(p+q-1) + 2$

Proof. *Consider the power graph ζ_G of the finite abelian group G . Using the number of vertex partition set and and edge partition set of the power graph ζ_G , we shall obtain results as follows,*

1. $M_1(\zeta_G)$

$$= \sum_{uv\in E(\zeta_G)} [d(u)+d(v)]$$
$$= p(q-1)+p(q-1))\tfrac{(q-1)(q-2)}{2} + (q(p-1)$$
$$+(pq-1)) + q(p-1)\tfrac{(p-1)(p-2)}{2} + ((pq-1)$$
$$\tfrac{(pq-q-p+2(pq-p-q+1))}{2} + ((p(q-1)+(pq-1))$$
$$(q-1)(pq-p-q+2) + (q(p-1)+(pq-1))$$
$$(p-1)(pq-p-q+2)$$
$$= (p^3 - 4p^2q + 10pq + pq^3 - 4pq^2 - 2p - 2q) +$$
$$(p^2q^2 - pq^2 - p^2q - 5pq + 2)(pq-p-q+2)$$
$$= p^3q^3 - 6p^2q^2 - 3pq - p^2 - q^2 - p - q + 5p^2q$$
$$+5pq^2 + 2$$
$$= |V|^3 - 6|V|^2 - |V| - (p+q)[(p+q) - 4|V|] + 2$$
$$= |G|^3 - 6|G|^2 - |G| - (p+q)[(p+q) - 4|G|] + 2$$

2. $M_2(\zeta_G)$

$$= \sum_{uv\in E(\zeta_G)} d(u)d(v)$$
$$= (p(q-1)\times p(q-1))\tfrac{(q-1)(q-2)}{2} + (q(p-1)$$
$$\times q(p-1))\tfrac{(p-1)(p-2)}{2} + ((pq-1)\times(pq-1))$$
$$\tfrac{(pq-q-p+2(pq-p-q+1))}{2} + ((p(q-1)\times(pq-1)))$$
$$(q-1)(pq-p-q+2)$$
$$+((q(p-1)\times(pq-1)))(p-1)(pq-p-q+2)$$
$$= \tfrac{1}{2}[p^4q^4 - 9p^3q^3 + 3p^2q^2 + 19pq + 8p^3q^2 + 8p^2q^3$$
$$-2p^3q - 2pq^3 - 13pq^2 - 13p^2q + 5p^2 + 5q^2 - 7p$$
$$-7q + 2]$$
$$= \tfrac{1}{2}[|V|^4 - 9|V|^3 + 11|V|^2(p+q) - |V|(13(p+q)+10)$$
$$+(p+q)[5(p+q)-7] + 2$$
$$= \tfrac{1}{2}[|G|^4 - 9|G|^3 + 11|G|^2(p+q) - |G|(13(p+q)+10)$$
$$+(p+q)[5(p+q)-7] + 2$$

3. $M_3(\zeta_G)$

$$= \sum_{uv\in E(\zeta_G)} d(u)-d(v)$$
$$= (p(q-1)-p(q-1))\tfrac{(q-1)(q-2)}{2} + (q(p-1)-$$
$$q(p-1))\tfrac{(p-1)(p-2)}{2} + ((pq-1)-(pq-1))$$
$$\tfrac{(pq-q-p+2(pq-p-q+1))}{2} + ((p(q-1)-(pq-1)))$$
$$(q-1)(pq-p-q+2) + ((q(p-1)-(pq-1)))$$
$$(p-1)(pq-p-q+2)$$
$$= p^2q^2 - 2p^2q - 2pq^2 + p^2 + q^2 + 5pq - 3p - 3q + 2$$
$$= |V|^2 - 2|V|(p+q+1) + 5|V| + 3(p+q-1) + 2$$
$$= |G|^2 - 2|G|(p+q+1) + +5|G| + 3(p+q-1) + 2$$

3.3 The Zagreb Co-Indices of the power graph of the finite abelian Group

In this section, we determine some types of Zagreb Co-indices and of its complement of power graph ζ_G of the finite abelian group G as follows,

Theorem 3.3. *Let ζ_G be the power graph of the finite abelian group. Then its Zagreb Co-indices ζ_G of and of its complements are given by*

1. $\overline{M_1}(\zeta_{\overline{G}}) = (p+q)(3+|G|) - 4|G| - p^2 - q^2 - 2$

2. $\overline{M_1}(\zeta_G) = 2|G|^2 + |G|[4-3p-3q] - p - q + p^2 + q^2$

3. $\overline{M_1}(\zeta_{\overline{G}}) = 2|G|^2 + |G|[4-3p-3q] - p - q + p^2 + q^2$

4. $M_2(\zeta_{\overline{G}})) = |G|^4 - 7|G|^3 + 10|G|^2 + 2|G|^2 +$

   . $2(p+q)[2|G|^2 - 6|G|) + 1] + 2(p^2+q^2) - 6$

5. $\overline{M_2}(\zeta_G)) = |G|^3 - 2|G|^2(p+q-2) - |G|[2(p+q+1]$

   . $+p^2+q^2]$

6. $\overline{M_2}(\zeta_{\overline{G}}) = \frac{1}{2}[10|G|(1-|G|) + (p+q)[7n+5]$

   . $+3(p^2+q^2)]$

Proof. *Consider the power graph ζ_G of the finite abelian group. Using the number of vertex partition set and edge partition set of the power graph ζ_G, we shall obtain results as follows,*

1. $M_1(\zeta_{\overline{G}})$

   $= M_1(\zeta_G) + |V|(|V|-1) - 4|E|(|V|-1)$

   $= M_1(\zeta_G) + pq(pq-1)^2 - 2(p^2q^2 - 3pq + 2p$

   $+2q-2)(pq-1)$

   $= M_1(\zeta_G) + p^3q^3 - 2p^2q^2 + pq - 2p^3q^3 + 4p^2q^2$

   $-4pq^2 - 4pq^2 - pq + 4p + 4q - 4$

   $= -4p^2q^2 + p^2q + q^2 - 3pq - p^2 - q^2 + 3p + 3Q - 2$

   $= -4|V|^2 + |V|(p+q) - 3|V| - p^2 - q^2 + 3(p+q) - 2$

   $= -4|G|^2 + |G|(p+q) - 3|G| - p^2 - q^2 + 3(p+q) - 2$

   $= (p+q)(3+|G|) - 4|G| - p^2 - q^2 - 2$

2. $\overline{M_1}(\zeta_G)$

   $= 2|E|(|V)| - 1) - M_1(\zeta_G)$

   $= p^2q^2 + 4pq - 3p^2q - 3pq^2 - p - q + p^2 + q^2$

   $= 2|V|^2 + |V|[4-3p-3q] - p - q + p^2 + q^2$

   $= 2|G|)^2 + |G|(4-3p-3q) - p - q + p^2 + q^2$

3. $M_2(\zeta_{\overline{G}})$

   $= \frac{1}{2} \times |V|(|V|-1)^3 - 3|E|(|V|-1)^2$

   $+2|E|^2 + \frac{2n-3}{2}M_1(\zeta_G) - M_2(\zeta_G)$

   $= |V|^4 - 7|V|^3 + 10|V|^2 + 2|V|^4$

   $+2(p+q)[2|V|^2 - 6|V| + 1] + 2(p^2+q^2) - 6$

   $= |G|^4 - 7|G|^3 + 10|G|^2 + 2|G|^2$

   $-6|G| + 1) + 2(p^2+q^2) - 6$

4. $\overline{M_2}(\zeta_G)$

   $= 2(|E|)^2 + \frac{1}{2}M_1(\zeta_G) - M_2(\zeta_G)$

   $= p^3q^3 + 4p^2q^2 - 2p^2q^2(p+q) - 4pq(p+q)$

   $+2pq(p^2+q^2) - 4pq$

   $= |V|^3 - 2|V|^2(p+q-2) - |V|[2(p+q+1) + p^2 + q^2]$

   $= |G|^3 - 2|G|^2(p+q-2) - |G|[2(p+q+1) + p^2 + q^2]$

5. $\overline{M_2}(\zeta_{\overline{G}})$

   $= 2|E|^2|V|^2 - (|V|-1)M_1(\zeta_G) + M_2(\zeta_G)$

   $= \frac{1}{2}10|V|(1-|V|) + (p+q)[7n+5] + 3(p^2+q^2)$

   $= \frac{1}{2}10|G|(1-|G|) + (p+q)[7n+5] + 3(p^2+q^2)$

4. Conclusion

In this paper, we partition the vertices and edges of the finite abelian group Z_{pq}, where $p < q$ and $p \neq q$ based on the degree of the vertices. The generators of the group is calulated from the vertex partition. Also zagreb indices are calculated with the help of power graph.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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