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Uniquely colourable graphs

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Abstract
A graph $G=(V,E)$ is uniquely colourable if the chromatic number $\delta\ (G)=n$ and every n-coloring of G induces the same partition of V. This paper studies concepts of uniquely colourable graph. Every uniquely k-colourable graph is k-connected. If G is a uniquely k-colourable graph then $\delta\ (G)$ =*k-1*. It was conjectured that for all graphs G of order at least two and all positive integers k there exist uniquely k-colourable graphs.

Keywords
Complete graph, bipartite graph, k-chromatic graph, uniquely k-colourable graph, k-colorable uniquely vertex k-colorable graphs.

AMS Subject Classification
05C85, 05C90.

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Contents

1. Introduction

There have been many generalizations of the notion of a vertex colouring of a graph. Some have attracted interest for their own sake, while others, for example, to hypergraphs, have yielded new results in chromatic theory. Most of the graph theoretic generalizations have revolved around colouring vertices so that the subgraph induced by each colour class has a given property P; properties of particular interest have included acyclicity, planarity and perfection [23,2,13,8,9,25,29]. Several authors [23,19, 22, 10,12] have proposed and investigated the generalized chromatic theory along these lines. In fact, in [14] it was shown that new results on hypergraph colourings related to criticality, unique colourability and complexity can be provided through generalized graph It seems natural (in the light of chromatic theory) to consider those graphs that have the least number of -G colourings. We define a graph F to be uniquely -Gk-colourable if it is -Gk-chromatic but, up to a permutation of colours, there is only one such -Gk-colouring. An example of a graph that is uniquely -$K_{1,3}$, 2-colourable is $K_{3,5}$. Uniquely k-colourable graphs have attracted considerable attention. Examples of such graphs are self-evident. Any connected bipartite graph is uniquely 2-colourable, and complete k-partite graphs are uniquely k-colourable. Thus the existence of such graphs for standard colourings is a non-issue.

2. Preliminaries

Definition 2.1. *Suppose that G is a k-chromatic graph. Then every k-colouring of G produces a partition of $V(G)$ into k independent subsets (colour classes). If every two k-colorings of G result in the same partition of $V(G)$ into color classes, then G is called uniquely k-colourable or simply uniquely colourable. Trivially, the complete graph Kn is uniquely colourable. Infact, every complete k-partite graph, $k=2$, is uniquely colourable.*

Note 2.2. *In graph theory, a uniquely colourable graph is a k-chromatic graph that has only one possible (proper) k-colouring up to permutation of the colures.*

2.1 Properties

Some properties of a uniquely k-colourable graph G with n vertices and m edges:

1. $m=(k-1)n-k(k-1)/2$.

3. Minimal imperfection

A minimal imperfect graph is a graph in which every subgraph is perfect. The deletion of any vertex from a minimal imperfect graph leaves a uniquely colourable subgraph.

3.1 Uniquely edge-colourable graph

A uniquely edge-colourable graph is a k-edge-chromatic graph that has only one possible (proper) k-edge-colouring up to permutation of the colours. The only uniquely 2-edge-colourable graphs are the paths and the cycles. For any k, the stars $K_{1,k}$ are uniquely k-edge-colourable graphs. Moreover, Wilson (1976) conjectured and Thomason (1978) proved that, when $k = 4$, they are also the only members in this family. However, there exist uniquely 3-edge-colourable graphs that do not fit into this classification, such as the graph of the triangular pyramid.

If a cubic graph is uniquely 3-edge-colourable, it must have exactly three Hamiltonian cycles, formed by the edges with two of its three colours, but some cubic graphs with only three Hamiltonian cycles are not uniquely 3-edge-colourable (belcastro & Haas 2015). Every simple planar cubic graph that is uniquely 3-edge-colourable contains a triangle (Fowler 1998), but W. T. Tutte (1976) observed that the generalized Petersen graph $G(9,2)$ is non-planar, triangle-free, and uniquely 3-edge-colourable. For many years it was the only known such graph, and it had been conjectured to be the only such graph (see Bollobas 1978 and Schwenk 1989) but now infinitely many triangle-free non-planar cubic uniquely 3-edge-colourable graphs .

3.2 Unique total colourability

A uniquely total colourable graph is a k-total-chromatic graph that has only one possible (proper) k-total-colouring up to permutation of the colors.
Empty graphs, paths, and cycles of length divisible by 3 are uniquely total colourable graphs.
Properties:

Some properties of a uniquely k-total-colourable graph G with n vertices:

1. $\chi"(G) = \Delta(G) + 1$ unless $G = K_2$
2. $\Delta(G) = 2\delta(G)$.
3. $\Delta(G) = n/2 + 1$.

Here $\chi"(G)$ is the total chromatic number; $\Delta(G)$, maximum degree; and $\delta(G)$, minimum degree.

Theorem 3.1. *Let G be a uniquely colourable graph. Let P be the chromatic partition for G. Let E be an independent k-set for G. $E \in P$ if and only if there exist a partition P_1 of $V - E$ such that*

1. *P_1 is unique*
2. *every set in P_1 is independent*
3. *$|P_1| = k - 1$ where $|P| = k$.*

Proof. Let G be a uniquely colorable graph. Let P be the chromatic partition for g. Let E be a k-set for G.
Let $|\text{P}| = k$. Assume that $E \in P$. Let $P = \{x_1, x_2, \dots, x_k\}$. Let $E \in x_i$. Any vertex in $x_j \in V - E$, $j = 1$ to k, $i \neq j$. Since P is the chromatic partition for G, $P_1 = \{x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k\}$ is a partition for $V -$E such that

1. P_1 is unique
2. every set in P_1 is independent
3. $|P_1| = k - 1$ where $|P| = k$.

Conversely, assume that there exists a partition P_1 for $V - E$ satisfying the conditions of the theorem. Let $P_1 = \{x_1, x_2, \dots, x_{k-1}\}$. Let $P = P_1 \cup \{E\}$.

1. $P_1 \cap E = \phi$
2. $P_1 \cup E = V(G)$
3. $x_i, E, i = 1$ to $k - 1$ are independent.

Hence P is a chromatic partition for G. □

Remark 3.2. *G has a chromatic partition P not containing any k-set if and only if either*

1. *G has no independent k-set*
2. *If G has an independent k-set then conditions of Theorem 3.1 fails.*

Proof. Let P be the chromatic partition not containing any k-set of G.In this case, it is obvious that

1. G has no independent k-set or
2. If G has an independent k-set then there exists no partition P_1 of $V - E$ satisfying the conditions of Theorem 3.1 (else if a partition exists for $V - E$ then the assumption that P does not contain any k-set fails).

Conversely, if the conditions of the remark satisfied, then P has no k-set. □

Theorem 3.3. *In every k-colouring of a uniquely k-colourable graph G, where $k \geq 2$, the sub graph of G induced by the union of every two colour classes of G is connected.*

Proof. Assume, to the contrary, that there exist two colour classes V_1 and V_2 in some k-colouring of G such that $H = G[V_1 \cap V_2]$ is disconnected. We may assume that the vertices in V_1 are coloured 1 and those in V_2 are coloured 2. Let H_1 and H_2 be two components of H. Interchanging the colours 1 and 2 of the vertices in H_1 produces a new partition of $V(G)$ into colour classes, producing a contradiction □

Theorem 3.4. *If G is a uniquely k-colourable graph, then $d(G) = k - l$.*

Proof. Much of the interest in uniquely colourable graphs has been directed towards planar graphs. Since every complete graph is uniquely colourable, each complete graph K_n, $1 \leq n \leq 4$, is a uniquely colourable planar graph. Indeed, each complete graph K_n, $1 \leq n \leq 4$ is a uniquely colourable maximal planar graph. Since the complete 3 partite graph $K_{2,2,2}$ (the graph of the octahedron) is also uniquely colorable, $K_{2,2,2}$ is a uniquely 3-colourable maximal planar graph. (see Figure 1). □

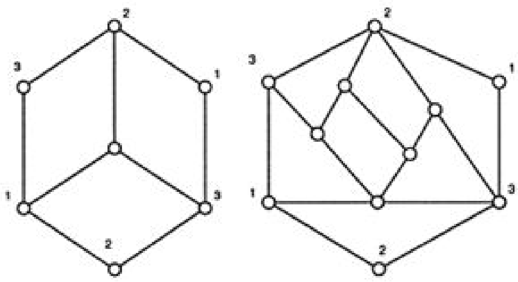

Figure 1. uniquely 3-colourable graph

Theorem 3.5. *A maximal planar graph G of order 3 or more has chromatic number 3 if and only if G is Eulerian.*

Proof. Let there be given a planar embedding of G . suppose first that G is not Eulerian. Then G contains a vertex v of odd degree $k = 3$. Let $M(v) = \{v1, v2, ..., vk, \}$, where $C = (v1, v2...vk, v1)$ is an odd cycle in G. Because v is adjacent to every vertex of C, it follows that $c(G) = 4$.
We verify the converse by induction on the order of maximal planar Eulerian graphs. If the order of G is 3, then $G = K_3$ and $\chi(G) = 3$. Assume that every maximal planar Eulerian graph of order k has chromatic number 3 for an integer $k = 3$ and let G be a maximal planar Eulerian graph of order $k+1$. Let there be given a planar embedding of G and let uw be an edge of G. Then uw is on the boundary of two (triangular) regions of G. Let x be the third vertex on the boundary of one of these regions and y the third vertex on the boundary of the other region. Suppose that $M(x) = \{u = x1, x2, \ldots, xk = w\}$ and $M(y) = \{u = y1, y2, \ldots, yl = w\}$.

Where k and l are even, such that $C = (x1, x2, \ldots, xk, x1)$ and $C' = (y1, y2, \ldots, yl, y1)$ are even cycles. Let G' be the graph obtained from G by deleting x, y, and uw from G and adding a new vertex z and joining z to every vertex of C and C'. Then G' is a maximal planar Eulerian graph of order k. By the induction hypothesis, $\chi(G') = 3$. According to Theorem 3.3, G' is uniquely colourable. Since z is adjacent to every vertex of C and C' we may assume that z is colored 1 and that the vertices of C and C' alternate in the colours 2 and 3. From the 3-coloring of G', a 3-colouring of G can be given where every vertex of $V(G) - \{x, y\}$ is assigned the same colour as in G' and x and y are colored 1.

On the other hand, Chartrand and Geller [1] showed that every uniquely 4-colourable planar graph must be maximal planar. □

4. k-partite and k-colourable

A k-colouring of a graph G, is a labeling of the vertices f : $V(G)!S$, where S is some set such that $jS_j = k$. Normally we think of the set S as a collection of k different colours, say $S = f$ red, blue, green, etc. g, or more abstractly as the positive integers $S = f1;2;:::;kg$. The labels are called colours. A k-colouring is proper if adjacent vertices are different colours. A graph is k-colorable if it has a proper k-colouring. The chromatic number $A(G)$ is the least positive integer k such that G is k-colourable.
You should notice that a graph is k-colourable if and only if it is k-partite. In other words, k-colourable and k-partite mean the same thing. You should convince yourself of this by determining the k different partite sets of a k-colourable graph and conversely determine a k-colouring of a k-partite graph.
In general it is not easy to determine the chromatic number of a graph or even if a graph is k-colourable for a given k.

Theorem 4.1. *If V is the chromatic partition for a uniquely colourable graph G , then every set in V is a dominating set.*

Proof. Let $V = \{x_1, x_2, \ldots, x_k\}$ be a chromatic partition for G. Assume that there exist some x_i, $i = 1$ to k such that x_i is not a dominating set then at least one vertex $u \in P(G)$, $u \in x_i$, such that u not adjacent to any vertex in x_i. Assume that $u \in x_j$, $j \neq i$, $V_1 = \{x_1, x_2, \ldots, x_{i-1}, x_i \cup \{u\}, x_{i+1}, \ldots, x_{j-1}, x_j \cup \{u\}, x_{j+1}, \ldots, x_k\}$ is a chromatic partition for G, a contradiction for our assumption that G is uniquely colourable. □

Theorem 4.2. *Let G be a uniquely colourable graph* $|P| = 2$ *if only if* $N(u) \in V - E$, $\forall u \in D$, $M(w) \in E$, $\forall w \in V - E$.

Proof. Let G be uniquely colourable and $|p| = 2 = \{x_1, x_2\}$ (say). If for some $u \in D$ a vertex $v \in V - E \ni v \in D$ then $P = \{x_1, x_2\}$ is a partition for G such that u, v belongs to some X_i, $i = 1, 2$, a contradiction to our assumption on P.
Similarly, if for some $w \in V - E$ there exist some $w \in V - E$ there exist some $x \in V - E$ such that $x \in N(w)$ then w, $x \in V - E$, w adjacent to v, w, v belongs to some x_i, $i = 1, 2$, a contradiction to our assumption on P.
Conversely, assume that for every $u \in D$, $M(u) \in V - E$ for all $w \in D$, $M(w) \in V - E$. If possible, assume that $|P| = 3 = \{x_1, x_2, x_3\}$ (say). Let one of x_i, $i = 1, 2, 3$ be a γ-set for G. Let $x_1 = E$, this means that x_2, $x_3 \in V - E$. By our assumption there exist no x, $y \in V - E$, $x \perp y$. So $x_2 \cup x_3$ is an independent set, which implies $P_1 = \{x_1, x_2 \cup x_3\}$ is a partition for G such that

1. x_1, $x_2 \cup x_3$ are independent
2. x_1 is a k-set for G
3. $x_2 \cup x_3 \in V - D$

That is, P_1 is a chromatic partition for G such that $|P_1| < |P|$, a contradiction to our assumption that P is a chromatic partition for G.

□

Remark 4.3. *For any tree, we know that $|P| = 2$, so if P is chromatic partition for T such that $|P| = 2 = \{x_1, x_2\}$ at least one of x_i is a k-set for T then by the above theorem, we conclude that the following statement T is a uniquely colourable tree if and only if for $u \in D$, $N(u) \in V - E$, $\forall w \in V - E$, $M(w) \in E$.*
By the Theorem 4.3, we conclude that
R1: If T is a uniquely colourable tree then

1. *every internal vertex is two dominated*
2. *if a pendant vertex $u \in D$, then for the support vertex v adjacent to u, u is the only leaf.*

Theorem 4.4. *Every uniquely k-colourable graph is $(k-1)$-connected.*

Proof. The result is trivial for $k = 1$ and, by Theorem 4.2, the result follows for $k = 2$ as well. Hence we may assume that $k = 3$. Let G be a uniquely k-colourable graph, where $k \geq 3$. If $G = Kk$, then G is $(k-1)$-connected; so we may assume, that G is not complete. Assume, to the contrary, that G is not $(k-1)$-connected. Hence there exists a vertex cut U of G with $U = k-2$.
Let there be given a k-colouring of G. Consequently, there are at least two colours, say 1 and 2, not used to colour any vertices of U. Let V_1 be the colour class consisting of the vertices coloured 1 and V_2 the set of the vertices coloured 2. By Theorem 4.2, $H = G[V_l \cup V_2]$ is connected. Hence H is a subgraph of some component G_1 of $G - U$. Let G 2 be another component of $G - U$. Assigning some vertex of G_2 the color 1 produces a new k-colouring of G that results in a new partition of $V(G)$ into color classes, contradicting our assumption that G is uniquely k-colourable. □

Theorem 4.5. *The subgraph induced by the union of any two colour classes in a k- colouring of a uniquely k-colourable graph is connected.*

Proof. Let C_1, C_2 be two colour classes in a k-colouring of the uniquely k-colourable graph G and $C_{1,2}$ be the subgraph induced by UC_2. If $C_{1,2}$ is disconnected each component of $C_{1,2}$ should contain vertices of both colours 1 and 2 in S we get a different k-colouring of G Contradicting the assumption. Hence $C_{1,2}$ is connected. □

Remark 4.6. *The converse of this result is not true.as illustrated by the following chromatic graph.*

Theorem 4.7. *The subgraph induced by the union of any h colour classes, $2 = k = k$, in a k-colouring of a uniquely k-colourable graph is $(h-1)$-connected.*

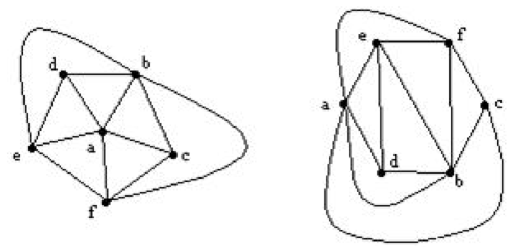

Figure 2. Non-uniquely 3-colourable graph

Proof. Such an induced subgraph is uniquely h-colourable. By Theorem 4.8 gives a necessary lower bound for the degrees of a uniquely k-colourable graph, the following result due to Bollobas (1978a) gives a sufficient lower bound, in terms of the number of vertices. □

Theorem 4.8. *If G is a k-colourable graph of order $n(k = 2)$ with $\delta(G) > n(3k-5)/(3k-2)$, then G is uniquely k-colourable.*

We omit the proof (by induction on k).
Harary, Hedetniemi and Robinson (1969) obtained several results on the construction of uniquely colourable graphs and proved that, for every $k = 3$, there is a uniquely k-colourable graph which as no induced K_k.
Uniquely colourable planar graphs have been studied by Chartrand and Gell (1969) and Aksionov (1977). In view of four-colour-theorem, only unique k-colourable planar graphs for $k = 2, 3$ or 4 need be considered and these have been studied fairly well

Theorem 4.9. *Let T be a k-uniquely colourable tree. Let $P = \{U_1, U_2\}$ be a δ-chromatic partition for T. H is generated from T by attaching a path P_1 at u where $v \in U(G)$. Let $\gamma(H) = \delta(T)$. H is δ-uniquely colourable if and only if $u \in V_1$.*

Proof. Assume that H is δ-uniquely colourable tree. There exist a δ-chromatic partition P_1 for H such that $P_1 = \{U_1, U_2\}$. Let D_1 be a k-set for H and D be a γ-uniquely colourable k-set for T. By assumption, $|D_1| = |D|$. Let v be a new pendant vertex attach at u to generate H. Either $u \in U_1$ or $v \in U_1$. If $v \in U_1$, then $D_1 - \{v\}$ is a k-set for T such that $|D| > |D_1 - \{v\}|$, a contradiction to our assumption that D is a k-set for T, implies $v \in U_1$. $P_2 = \{P_1 - \{v\}\} = \{U_1, U_2 - \{v\}\}$ is a δ-chromatic partition for T. $P_2 = P$ since T is γ-uniquely colourable tree.
Conversely, assume that $uv \in V$, we have to prove that H is γ-uniquely colourable tree. D is a k-set for H and $P_3 = \{P \cup \{u\}\} = \{U_1, U_2 \cup \{u\}\}$ is a chromatic partition for H such that

1. $U_1 \in D$.
2. $U_2 \in U - E$.
3. $M(u) \in U - E$, for all $v \in E$.

4. $M(w) \in E$, for all $w \in U - E$. implies H is δ-uniquely colourable tree.

□

Note 4.10. *Theorem 4.4 states that, H is δ-uniquely colourable tree if and only if $u \in V_1$. If u is any vertex in H which is a good vertex but $u \in V_1$, then the resulting graph H need not be uniquely colourable. For example, consider the graph G in Figure 3.*

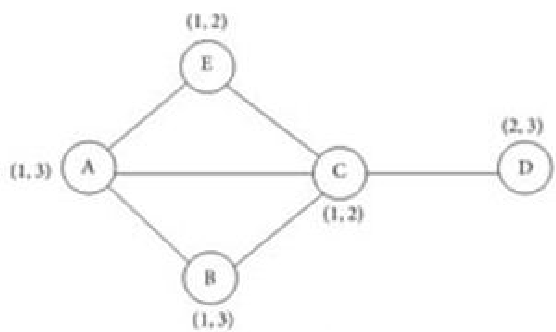

Figure 3. The list of graph

Theorem 4.11. *Let T be a k-uniquely colourable tree. Let $P = \{V_1, V_2\}$ be a γ-chromatic partition for T. H is generated from T by attaching a path P_1 at u where $u \in V(G)$. Let $\gamma(H) = \gamma(T) + 1$. H is k-uniquely colourable if and only if $u \in$ bad.*

Proof. Assume that H is k-uniquely colourable tree. There exist a k-chromatic partition P_1 for H such that $P_1 = \{V_1, V_2\}$. Let D_1 be a γ-set for H and D be a γ-uniquely colourable k-set for T. Let v be a new pendant vertex attach at u to generate. u is a bad vertex with the vertex to T else if u is good with respect to T, then D itself is a γ-set for H, a contradiction to our assumption that $\gamma(H) = \gamma(T) + 1$.
Conversely, assume that u is bad with respect to T. Since $\gamma(H) = \gamma(T) + 1$, let $D_1 = D \cup \{v\}$ be a γ-set for H. Since T is γ-uniquely colourable, $P_1 = P \cup \{v\} = \{V_1 \cup \{v\}, V_2\} = \{V_3, V_2\}$ is a chromatic polynomial for H such that

1. $v \in V_3$
2. $u \in V_2$
3. $N(u) \in V - D_1$, for all $u \in D_1$
4. $N(w) \in D_1$, for all $w \in V - D_1$

implies D_1 is a γ-uniquely colourable γ-set for H and hence H is γ-uniquely colourable tree. □

Theorem 4.12. *Let T be a γ-uniquely colourable tree. Let $P = \{V_1, V_2\}$ be a γ-chromatic partition for T. H is generated from T by attaching a path P_2 at u where $u \in V(G)$. Let $\gamma(H) = \gamma(T) + 1$. H is γ-uniquely colourable if and only if u is not selfish with respect to T.*

Proof. Assume that H is γ-uniquely colourable tree. H is generated from T by attaching a path P_2 to u. There exist a γ-chromatic partition P_1 for H such that $P_1 = \{V_1, V_2\}$. Let D_1 be a k-set for H and D be a γ-uniquely colourable k-set for T. Let the vertex adjacent to u be v and w be the vertex adjacent to v. If possible, assume that u is selfish with respect to T. Then $D_1 = D - \{u\} \cup \{v\}$ is a k-set for H. Since H is uniquely colourable there exist a γ-uniquely colourable k-set D_2 for H and a γ-chromatic partition $P_1 = \{V_3, V_4\}$ for H. Either $v \in V_3$ or $w \in V_3$ (since w is pendant). $D_3 = D_2$ - $\{v\}$ is a γ-set for T such that $|D_3| < |D_2|$, a contradiction to our assumption that D_2 is a γ-set for H. If $v \in V_3$, then $u \in V - D_2$ and u is 2-dominated with respect to D_2. If $w \in V_3$, then $v \in V - D_2$ and $u \in V_3$. $D_3 = D_2 - \{w\}$ is a γ-set for T such that $|D_3| < |D_2|$, a contradiction to our assumption that D_2 is a k-set for H.
Conversely, assume that u is not selfish with respect to T. We know that T is uniquely colorable and $P = \{V_1, V_2\}$ is a γ-chromatic partition for T. $D_4 = D \cup \{w\}$ is a γ-set for H (since $\gamma(H) = \gamma(T) + 1$). Also $P_2 = \{P \cup \{w\}\} = \{V_1 \cup \{w\}, V_2\} = \{V_5, V_2\}$ is a chromatic partition for H such that

1. $N(u) \in V - D_4$, for all $u \in D_4$ and
2. $N(w) \in D$, for all $w \in V - D_4$ implies H is uniquely colorable.

□

5. Conclusion

This paper contributes the necessary and sufficient condition, of k - uniquely colorable graphs and some of theorems on uniquely colourable graph is proved.

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