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Lagrange's Quadratic Functional Equation Connected with Homomorphisms and Derivations on Lie C^*-algebras: Direct and Fixed Point Methods

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Abstract

In this paper, we obtain the general solution in vector space and the generalized Ulam-Hyers stability of Lagrange's quadratic functional equation of the form

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} f(y_i)\right) = f\left(\sum_{i=1}^{n} x_i y_i\right) + \sum_{1\le i<j\le n} f\left(x_i y_j - x_j y_i\right)$$

where n is a positive integer on Lie C^*-algebras using direct and fixed point methods. An application of this functional equation is also studied.

Keywords: Quadratic functional equation, Generalized Ulam-Hyers stability, Lie -algebra, Fixed point method.

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1 Introduction and Preliminaries

The study of stability problems for functional equations is related to a question of Ulam [27] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [13]. It was further generalized and excellent results obtained by number of authors [2, 12, 23, 24, 25]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

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is said to be **quadratic functional equation** because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.1).

The stability problems of several quadratic functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem M. Arunkumar et al., [3], I.S. Chang, H.M. Kim [8], S. Czerwik [9], M. Eshaghi Gordji, H. Khodaei [10], Y.H. Bae, K. W. Jun [14], S.M. Jung [15] and PL. Kannappan [16].

Now, we give some definitions which helps to investigate the stability results in Lie C^*-algebra.

A C^*-algebra C, endowed with Lie product $[x,y] := \frac{xy-yx}{2}$ on C, is called a Lie C^*-algebra ([20], [21], [22]).

Definition 1.1. *[11] Let $\mathcal{A},\mathcal{B}$ be two algebras. A mapping $f : \mathcal{A} \to \mathcal{B}$ is called a quadratic homomorphism if f is a quadratic mapping satisfying $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}$. For instance, let $\mathcal{A}$ be commutative. Then the mapping $f : \mathcal{A} \to \mathcal{A}$, defined by $f(x) = x^2 (x \in \mathcal{A})$, is a quadratic homomorphism.*

Definition 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Lie C^*-algebras. A $\mathbb{C}$-linear mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ is called a Lie C^* quadratic homomorphism if $\mathcal{H}([x,y]) = [\mathcal{H}(x), \mathcal{H}(y)]$ for all $x, y \in \mathcal{A}$.*

Definition 1.3. *[11] Let $\mathcal{A}$ be a algebra. A mapping $f : \mathcal{A} \to \mathcal{A}$ is called a quadratic derivation if f is a quadratic mapping satisfying $f(xy) = x^2 f(y) + f(x)y^2$ for all $x, y \in \mathcal{A}$.*

Definition 1.4. *Let $\mathcal{A}$ be a Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is called a Lie C^* quadratic derivation if $\mathcal{D}([x,y]) = \left[\mathcal{D}(x), y^2\right] + \left[x^2, \mathcal{D}(y)\right]$ for all $x, y \in \mathcal{A}$.*

For more details one can refer to [6], [7], [18], [26] and [28].

Recently, M. Arunkumar and S. Karthikeyan [4] obtained the general solution and the generalized Ulam-Hyers stability of Brahmagupta quadratic functional equations of the form

$$(f(x_1) + nf(x_2))\,(f(x_3) + nf(x_4)) = f\,(x_1x_3 \pm nx_2x_4) + nf\,(x_1x_4 \mp x_2x_3) \tag{1.2}$$

on non-Archimedean Banach algebras using direct and fixed point methods.

In this paper, the authors proved the general solution in vector space and established the generalized Ulam-Hyers stability of Lagrange's quadratic functional equation of the form

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} f(y_i)\right) = f\left(\sum_{i=1}^{n} x_i y_i\right) + \sum_{1\le i<j\le n} f\left(x_i y_j - x_j y_i\right) \tag{1.3}$$

where n is a positive integer in Lie C^*-algebras using direct and fixed point methods. An application of this functional equation is also studied.

2 General Solution of the Functional Equation (1.3)

In this section, the authors investigate the general solution of the Lagrange's quadratic functional equation (1.3). Throughout this section let us consider X and Y be real vector spaces.

Theorem 2.1. *Let X and Y be real vector spaces. If the mapping $f : X \to Y$ satisfies the functional equation (1.3) for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in X$ then $f : X \to Y$ satisfying the functional equation (1.1) for all $x, y \in X$.*

Proof. Setting $(x_1,x_2,x_3,\cdots,x_n)$ and $(y_1,y_2,y_3,\cdots,y_n)$ by $(x_1,x_2,0,\cdots,0)$ and $(y_1,y_2,0,\cdots,0)$ in 1.3, we get

$$(f(x_1)+f(x_2))\,(f(y_1)+f(y_2)) = f\,(x_1y_1+x_2y_2)+f\,(x_1y_2-x_2y_1) \tag{2.1}$$

for all $x_1,x_2,y_1,y_2 \in X$. Replacing (x_1,x_2,y_1,y_2) by $(\sqrt{x},0,\sqrt{x},0)$ in (2.1), we obtain

$$\left(f\left(\sqrt{x}\right)\right)^2 = f(x) \tag{2.2}$$

for all $x \in X$. Setting (x_1,x_2,y_1,y_2) by $(\sqrt{x},0,\sqrt{y},0)$ in (2.1), we get

$$f\left(\sqrt{x}\right)f\left(\sqrt{y}\right) = f\left(\sqrt{x}\sqrt{y}\right) \tag{2.3}$$

for all $x,y \in X$. Replacing (x_1,x_2,y_1,y_2) by $(\sqrt{x},\sqrt{x},\sqrt{x},\sqrt{x})$ in (2.1) and using (2.2), we arrive

$$4f(x) = f(2x) \tag{2.4}$$

for all $x \in X$. Letting (x_1,x_2,y_1,y_2) by $(x,0,x,0)$ in (2.1), we obtain

$$f(x) = \sqrt{f(x^2)} \tag{2.5}$$

for all $x \in X$. Replacing x by $-x$ in (2.5), we get $f(-x) = f(x)$ is an even function. Setting (x_1,x_2,y_1,y_2) by $(\sqrt{x},\sqrt{y},\sqrt{x},\sqrt{y})$ in (2.1) and using (2.2), we get

$$f(x)+f(y)+2f\left(\sqrt{x}\right)f\left(\sqrt{y}\right) = f(x+y) \tag{2.6}$$

for all $x,y \in X$. Letting (x_1,x_2,y_1,y_2) by $(\sqrt{x},\sqrt{y},\sqrt{y},\sqrt{x})$ in (2.1) and using (2.2), (2.3) and (2.4) , we obtain

$$f(x)+f(y)-2f\left(\sqrt{x}\right)f\left(\sqrt{y}\right) = f(x-y) \tag{2.7}$$

for all $x,y \in X$. Adding (2.6) and (2.7), we derive (1.1). □

3 Stability Results: Direct Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.3).

Throughout this section, let $\mathcal{A}$ be a Lie C^*-algebra with norm $\|\,.\,\|_A$ and $\mathcal{B}$ be a Lie C^*-algebra with norm $\|\,.\,\|_B$. Define a mapping $F:\mathcal{A}\to\mathcal{B}$ by

$$F(x_1,y_1,x_2,y_2,\cdots,x_n,y_n) = \left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} f(y_i)\right) - f\left(\sum_{i=1}^{n} x_iy_i\right) - \sum_{1\le i<j\le n} f\left(x_iy_j - x_jy_i\right)$$

for all $x_1,y_1,x_2,y_2,\cdots,x_n,y_n \in \mathcal{A}$.

Theorem 3.1. *Let $j\in\{-1,1\}$. Let $\alpha:\mathcal{A}^{2n}\to[0,\infty)$ be a function such that*

$$\sum_{k=0}^{\infty}\frac{\alpha\left(n^{kj}x_1,n^{kj}y_1,n^{kj}x_2,n^{kj}y_2,\cdots,n^{kj}x_n,n^{kj}y_n\right)}{n^{2kj}} \text{ converges to } \mathbb{R} \text{ and}$$

$$\lim_{k\to\infty}\frac{\alpha\left(n^{kj}x_1,n^{kj}y_1,n^{kj}x_2,n^{kj}y_2,\cdots,n^{kj}x_n,n^{kj}y_n\right)}{n^{2kj}} < \infty \tag{3.1}$$

for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$ *and let* $f : \mathcal{A} \to \mathcal{B}$ *be a function satisfies the inequality*

$$\|F(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)\|_B \leq \alpha\,(x_1, y_1, x_2, y_2, \cdots, x_n, y_n) \tag{3.2}$$

for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$ *and*

$$\|f\,[(x,y)] - [f(x), f(y)]\|_B \leq \alpha\,(x, y, 0, 0, \cdots, 0, 0) \tag{3.3}$$

for all $x, y \in \mathcal{A}$. *Then there exists a unique quadratic homomorphism* $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(x) - \mathcal{H}(x)\|_B \leq \frac{1}{n^2} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\sqrt{n}\right)^{ij}\sqrt{x}, \left(\sqrt{n}\right)^{ij}\sqrt{x}, \cdots, \left(\sqrt{n}\right)^{ij}\sqrt{x}\right)}{n^{2ij}} \tag{3.4}$$

for all $x \in \mathcal{A}$. *The mapping* $\mathcal{H}(x)$ *is defined by*

$$\mathcal{H}(x) = \lim_{k\to\infty} \frac{f(n^{kj}x)}{n^{2kj}} \tag{3.5}$$

for all $x \in \mathcal{A}$.

Proof. Assume $j = 1$. Replacing $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ by $\left(\sqrt{x}, \sqrt{x}, \sqrt{x}, \sqrt{x}, \cdots, \sqrt{x}, \sqrt{x}\right)$ and dividing by n^2 in (1.3), we get

$$\left\|\frac{f(nx)}{n^2} - f(x)\right\|_B \leq \frac{1}{n^2}\alpha\left(\sqrt{x}, \sqrt{x}, \cdots, \sqrt{x}\right) \tag{3.6}$$

for all $x \in \mathcal{A}$. Replacing x by nx in (3.6) and divided by n^2, we get

$$\left\|\frac{f(n^2x)}{n^4} - \frac{f(nx)}{n^2}\right\|_B \leq \frac{\alpha\left(\sqrt{nx}, \sqrt{nx}, \cdots, \sqrt{nx}\right)}{n^4} \tag{3.7}$$

for all $x \in \mathcal{A}$. Combining (3.6) and (3.7), we obtain

$$\left\|\frac{f(n^2x)}{n^4} - f(x)\right\|_B \leq \frac{1}{n^2}\left(\alpha\left(\sqrt{x}, \sqrt{x}, \cdots, \sqrt{x}\right) + \frac{\alpha\left(\sqrt{nx}, \sqrt{nx}, \cdots, \sqrt{nx}\right)}{n^2}\right) \tag{3.8}$$

for all $x \in \mathcal{A}$. Using induction on a positive integer k , we obtain that

$$\begin{aligned}\left\|\frac{f(n^kx)}{n^{2k}} - f(x)\right\|_B &\leq \frac{1}{n^2}\sum_{i=0}^{k-1} \frac{\alpha\left(\sqrt{n^ix}, \sqrt{n^ix} \cdots, \sqrt{n^ix}\right)}{n^{2i}} \\ &\leq \frac{1}{n^2}\sum_{i=0}^{\infty} \frac{\alpha\left(\sqrt{n^ix}, \sqrt{n^ix}, \cdots, \sqrt{n^ix}\right)}{n^{2i}}\end{aligned} \tag{3.9}$$

for all $x \in \mathcal{A}$. In order to prove the convergence of the sequence $\left\{\frac{f(n^kx)}{n^{2k}}\right\}$, replace x by n^mx and divided by n^{2m} in (3.9), for any $m, k > 0$, we arrive

$$\begin{aligned}\left\|\frac{f(n^kn^mx)}{n^{2k+2m}} - \frac{f(n^mx)}{n^{2m}}\right\|_B &\leq \frac{1}{n^2}\sum_{i=0}^{k-1} \frac{\alpha\left(\sqrt{n^{i+m}x}, \sqrt{n^{i+m}x}, \cdots, \sqrt{n^{i+m}x}\right)}{n^{2(i+m)}} \\ &\leq \frac{1}{n^2}\sum_{i=0}^{\infty} \frac{\alpha\left(\sqrt{n^{i+m}x}, \sqrt{n^{i+m}x}, \cdots, \sqrt{n^{i+m}x}\right)}{n^{2(i+m)}}\end{aligned} \tag{3.10}$$

for all $x \in \mathcal{A}$. Since the right hand side of the inequality (3.10) tends to 0 as $m \to \infty$, the sequence $\left\{\frac{f(n^kx)}{n^{2k}}\right\}$ is a Cauchy sequence. Since $\mathcal{B}$ is complete, there exists a mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{H}(x) = \lim_{k\to\infty} \frac{f(n^kx)}{n^{2k}}, \quad \forall\ x \in \mathcal{A}.$$

Letting $k \to \infty$ in (3.9), we see that (3.4) holds for all $x \in \mathcal{A}$. Now we need to prove $\mathcal{H}$ satisfies (1.3), replacing $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ by $(n^k x_1, n^k y_1, \cdots, n^k x_n, n^k y_n)$ and divided by n^{2k} in (3.2), we arrive

$$\frac{1}{n^{2k}} \left\| F\left(n^k x_1, n^k y_1, \cdots, n^k x_n, n^k y_n\right) \right\|_B \leq \frac{\alpha\left(n^k x_1, n^k y_1, \cdots, n^k x_n, n^k y_n\right)}{n^{2k}}$$

for all $x_1, y_1, \cdots, x_n, y_n \in \mathcal{A}$. Letting $k \to \infty$ in the above inequalities, we arrive

$$\left\| \mathcal{H}\left(n^k x_1, n^k y_1, n^k x_2, n^k y_2, \cdots, n^k x_n, n^k y_n\right) \right\|_B = 0.$$

Hence $\mathcal{H}$ satisfies (1.3) for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$. This shows that $\mathcal{H}$ is quadratic. Also

$$\begin{aligned}\|\mathcal{H}([x,y]) - [\mathcal{H}(x), \mathcal{H}(y)]\|_B &= \lim_{k\to\infty} \frac{1}{n^{4k}} \left\| f\left(n^{2k}[x,y]\right) - \left[f\left(n^k x\right), f\left(n^k y\right)\right] \right\|_B \\ &\leq \lim_{k\to\infty} \frac{1}{n^{4k}} \alpha\left(n^k x, n^k y, 0, \cdots, 0\right) = 0\end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, $\mathcal{H}$ is a quadratic homomorphism. In order to prove $\mathcal{H}$ is unique, let $\mathcal{H}'(x)$ be another quadratic mapping satisfying (3.4) and (1.3). Then

$$\begin{aligned}\left\|\mathcal{H}(x) - \mathcal{H}'(x)\right\|_B &= \frac{1}{n^{2k}} \left\| \mathcal{H}(n^k x) - \mathcal{H}'(n^k x) \right\|_B \\ &\leq \frac{1}{n^{2k}} \left\{ \left\| \mathcal{H}(n^k x) - f(n^k x) \right\|_B + \left\| f(n^k x) - \mathcal{H}'(n^k x) \right\|_B \right\} \\ &\leq \frac{2}{n^2} \sum_{i=0}^{\infty} \frac{\alpha\left(\sqrt{n^{i+k}x}, \sqrt{n^{i+k}x}, \cdots, \sqrt{n^{i+k}x}\right)}{n^{2(k+i)}} \\ &\to 0 \quad as \ \ k \to \infty\end{aligned}$$

for all $x \in \mathcal{A}$. Hence $\mathcal{H}$ is unique.

For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.3).

Corollary 3.1. *Let λ and s be nonnegative real numbers. If a function $f : \mathcal{A} \to \mathcal{B}$ satisfies the inequality*

$$\|F(x_1, y_1, \cdots, x_n, y_n)\|_B \quad \leq \begin{cases} \lambda, & \\ \lambda \sum\limits_{i=1}^{n} \left\{ ||x_i||_A^s + ||y_i||_A^s \right\}, & s \neq 4; \\ \lambda \left\{ \prod\limits_{i=1}^{n} ||x_i||_A^s ||y_i||_A^s + \sum\limits_{i=1}^{n} \left(||x_i||_A^{2ns} + ||y_i||_A^{2ns} \right) \right\}, & s \neq \frac{2}{n}; \end{cases} \tag{3.11}$$

for all $x_1, y_1, \cdots, x_n, y_n \in \mathcal{A}$ and

$$\|f[(x,y)] - [f(x), f(y)]\|_B \leq \begin{cases} \lambda, \\ \lambda \left\{ ||x||_A^s + ||y||_A^s \right\}, \\ \lambda \left\{ ||x||_A^s ||y||_A^s + \left(||x||_A^{2s} + ||y||_A^{2s} \right) \right\}, \end{cases} \tag{3.12}$$

for all $x, y \in \mathcal{A}$. Then there exists a unique quadratic homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_B \leq \begin{cases} \dfrac{\lambda}{|n^2 - 1|}, \\ \dfrac{2\lambda ||x||_A^{s/2}}{|n^2 - n^{s/2}|}, \\ \dfrac{2\lambda ||x||_A^{ns}}{|n^2 - n^{ns}|} \end{cases} \tag{3.13}$$

for all $x \in \mathcal{A}$.

Theorem 3.2. *Let $j \in \{-1,1\}$. Let $\alpha : \mathcal{A}^{2n} \to [0,\infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(n^{kj}x_1, n^{kj}y_1, n^{kj}x_2, n^{kj}y_2, \cdots, n^{kj}x_n, n^{kj}y_n\right)}{n^{2kj}} \text{ converges to } \mathbb{R} \text{ and}$$

$$\lim_{k\to\infty} \frac{\alpha\left(n^{kj}x_1, n^{kj}y_1, n^{kj}x_2, n^{kj}y_2, \cdots, n^{kj}x_n, n^{kj}y_n\right)}{n^{2kj}} < \infty \tag{3.14}$$

for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$ and let $f : \mathcal{A} \to \mathcal{A}$ be a function satisfies the inequality

$$\|F(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)\|_B \le \alpha(x_1, y_1, x_2, y_2, \cdots, x_n, y_n) \tag{3.15}$$

for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$ and

$$\left\| f([x,y]) - \left[x^2, f(y)\right] - \left[f(x), y^2\right] \right\|_B \le \alpha(x, y, 0, \cdots, 0) \tag{3.16}$$

for all $x, y \in \mathcal{A}$. Then there exists a unique quadratic derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \mathcal{D}(x)\|_B \le \frac{1}{n^2} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\sqrt{n}\right)^{ij}\sqrt{x}, \left(\sqrt{n}\right)^{ij}\sqrt{x}, \cdots, \left(\sqrt{n}\right)^{ij}\sqrt{x}\right)}{n^{2ij}} \tag{3.17}$$

for all $x \in \mathcal{A}$. The mapping $\mathcal{D}(x)$ is defined by

$$\mathcal{D}(x) = \lim_{k\to\infty} \frac{f(n^{kj}x)}{\ell^{2kj}} \tag{3.18}$$

for all $x \in \mathcal{A}$.

Proof. Assume $j = 1$. By the same reasoning as that in the proof of the Theorem 3.1, there exist a unique quadratic mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying (3.17). The mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ given by $\mathcal{D}(x) = \lim_{k\to\infty} \frac{f(n^k x)}{n^{2k}}$. It follows from (3.15) that

$$\begin{aligned}
&\left\| \mathcal{D}([x,y]) - \left[x^2 \mathcal{D}(y)\right] - \left[\mathcal{D}(x) y^2\right] \right\|_B \\
&= \lim_{k\to\infty} \frac{1}{n^{4kj}} \left\| f\left(n^{2k}[x,y]\right) - \left[\left(n^k x\right)^2, f\left(n^k y\right)\right] - \left[f\left(n^k x\right), \left(n^k y\right)^2\right] \right\|_B \\
&\le \lim_{k\to\infty} \frac{1}{n^{4k}} \alpha\left(n^k x, n^k y, 0, \cdots, 0\right) = 0
\end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is a quadratic derivation satisfying (3.17). □

The following corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.3).

Corollary 3.2. *Let λ and s be nonnegative real numbers. If a function $f : \mathcal{A} \to \mathcal{A}$ satisfies the inequality*

$$\|F(x_1, y_1, \cdots, x_n, y_n)\|_B \le \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \left\{ ||x_i||_A^s + ||y_i||_A^s \right\}, & s \neq 4; \\ \lambda \left\{ \prod_{i=1}^{n} ||x_i||_A^s ||y_i||_A^s + \sum_{i=1}^{n} \left(||x_i||_A^{2ns} + ||y_i||_A^{2ns} \right) \right\}, & s \neq \frac{2}{n}; \end{cases} \tag{3.19}$$

for all $x_1, y_1, \cdots, x_n, y_n \in \mathcal{A}$ and

$$\left\| f([x,y]) - \left[x^2, f(y)\right] - \left[f(x), y^2\right] \right\|_B \le \begin{cases} \lambda, \\ \lambda \left\{ ||x||_A^s + ||y||_A^s \right\}, \\ \lambda \left\{ ||x||_A^s ||y||_A^s + \left(||x||_A^{2ns} + ||y||_A^{2ns} \right) \right\}, \end{cases} \tag{3.20}$$

for all $x, y \in \mathcal{A}$. Then there exists a unique quadratic derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \mathcal{D}(x)\|_B \leq \begin{cases} \dfrac{\lambda}{|n^2 - 1|}, \\ \dfrac{2\lambda||x||_A^{s/2}}{|n^2 - n^{s/2}|}, \\ \dfrac{2\lambda||x||_A^{ns}}{|n^2 - n^{ns}|} \end{cases} \tag{3.21}$$

for all $x \in \mathcal{A}$.

4 Stability Results: Fixed Point Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.3) in Lie C^*-algebras using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let $(\mathcal{A}, d)$ be a complete metric space and consider a mapping $T : \mathcal{A} \to \mathcal{A}$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in \mathcal{A}$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} \ d(T^n x, T^{n+1}x), \forall \ n \geq 0, \forall \ x \in \mathcal{A}$;

(A4) $d(x, x^*) \leq \frac{1}{1-L} \ d(x, x^*), \forall \ x \in \mathcal{A}$.

Theorem 4.2. *[19](The alternative of fixed point) Suppose that for a complete generalized metric space $(\mathcal{A}, d)$ and a strictly contractive mapping $T : \mathcal{A} \to \mathcal{A}$ with Lipschitz constant L. Then, for each given element $x \in \mathcal{A}$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall \ n \geq 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii)The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in \mathcal{A} : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} \ d(y, Ty)$ for all $y \in Y$.

Theorem 4.3. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there exist a function $\varphi : \mathcal{A}^{2n} \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \varphi(\mu_i^k x_1, \mu_i^k y_1, \mu_i^k x_2, \mu_i^k y_2, \cdots, \mu_i^k x_n, \mu_i^k y_n) = 0 \tag{4.1}$$

where $\mu_i = n$ if $i = 0$ and $\mu_i = \dfrac{1}{n}$ if $i = 1$ such that the functional inequality with

$$\|F(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)\|_B \leq \varphi(x_1, y_1, x_2, y_2, \cdots, x_n, y_n) \tag{4.2}$$

for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$ *and*

$$\|f([x,y]) - [f(x), f(y)]\|_B \leq \varphi(x, y, 0, \cdots, 0) \tag{4.3}$$

for all $x, y \in \mathcal{A}$. *If there exists* $L = L(i) < 1$ *such that the function*

$$x \to \gamma(x) = \varphi\left(\sqrt{\frac{x}{n}}, \sqrt{\frac{x}{n}}, \cdots, \sqrt{\frac{x}{n}}\right), \tag{4.4}$$

has the property

$$\gamma(x) = L\frac{1}{\mu_i^2}\gamma(\mu_i x). \tag{4.5}$$

for all $x \in \mathcal{A}$. *Then there exists a unique quadratic homomorphism* $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ *satisfying the functional equation (1.3) and*

$$\| f(x) - \mathcal{H}(x) \|_B \leq \frac{L^{1-i}}{1-L}\gamma(x) \tag{4.6}$$

for all $x \in A$.

Proof. Consider the set $\Omega = \{p/p : \mathcal{A} \to \mathcal{B},\ p(0) = 0\}$ and introduce the generalized metric on Ω,

$$d(p,q) = d_\gamma(p,q) = \inf\{K \in (0,\infty) : \|p(x) - q(x)\|_B \leq K\gamma(x), x \in \mathcal{A}\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by $Tp(x) = \frac{1}{\mu_i^2}p(\mu_i x)$, for all $x \in \mathcal{A}$. One can show that $d(Tp, Tq) \leq Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
Replacing $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ by $(\sqrt{x}, \sqrt{x}, \sqrt{x}, \sqrt{x}, \cdots, \sqrt{x}, \sqrt{x})$ in (1.3), we get,

$$\left\|n^2 f(x) - f(nx)\right\|_B \leq \varphi\left(\sqrt{x}, \sqrt{x}, \cdots, \sqrt{x}\right) \tag{4.7}$$

Hence from the above inequality, we have

$$\left\|f(x) - \frac{f(nx)}{n^2}\right\|_B \leq \frac{1}{n^2}\varphi\left(\sqrt{x}, \sqrt{x}, \cdots, \sqrt{x}\right) \tag{4.8}$$

for all $x \in \mathcal{A}$. Using (4.4) and (4.5) for the case $i = 0$, it reduces to

$$\left\|f(x) - \frac{f(nx)}{n^2}\right\|_B \leq \frac{1}{n^2}\gamma(x)$$

for all $x \in \mathcal{A}$, i.e., $d_\varphi(f, Tf) \leq L \Rightarrow d(f, Tf) \leq L \leq L^1 < \infty$. Again replacing x by $\frac{x}{n}$ in (4.7), we get,

$$\left\|n^2 f\left(\frac{x}{n}\right) - f(x)\right\|_B \leq \varphi\left(\sqrt{\frac{x}{n}}, \sqrt{\frac{x}{n}}, \cdots, \sqrt{\frac{x}{n}}\right) \tag{4.9}$$

for all $x \in \mathcal{A}$. Using (4.4) and (4.5) for the case $i = 1$ it reduces to

$$\left\|f(x) - n^2 f\left(\frac{x}{n}\right)\right\|_B \leq \gamma(x)$$

for all $x \in \mathcal{A}$, i.e., $d_\varphi(f, Tf) \leq 1 \Rightarrow d(f, Tf) \leq 1 \leq L^0 < \infty$.

In both cases, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore $(A1)$ holds.

By $(A2)$, it follows that there exists a fixed point $\mathcal{H}$ of T in Ω such that

$$\mathcal{H}(x) = \lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \left(f(\mu_i^k x) \right) \tag{4.10}$$

for all $x \in \mathcal{A}$.

To prove $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ is quadratic. Replacing $(x_1, y_1, \cdots, x_n, y_n)$ by $\left(\mu_i^k x_1, \mu_i^k y_1, \cdots, \mu_i^k x_n, \mu_i^k y_n\right)$ in (4.2) and dividing by μ_i^k, it follows from (4.1) that

$$\begin{aligned}\|\mathcal{H}(x_1, y_1, \cdots, x_n, y_n)\|_B &= \lim_{k\to\infty} \frac{\left\|F(\mu_i^k x_1, \mu_i^k y_1, \cdots, \mu_i^k x_n, \mu_i^k y_n)\right\|_B}{\mu_i^{2k}} \\ &\leq \lim_{k\to\infty} \frac{\varphi(\mu_i^k x_1, \mu_i^k y_1, \cdots, \mu_i^k x_n, \mu_i^k y_n)}{\mu_i^{2k}} = 0\end{aligned}$$

for all $x_1, y_1, \cdots, x_n, y_n \in \mathcal{A}$. i.e., $\mathcal{H}$ satisfies the functional equation (1.3). Also,

$$\begin{aligned}\|\mathcal{H}([x,y]) - [\mathcal{H}(x), \mathcal{H}(y)]\|_B &= \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \left\| f\left(\mu_i^{2k}[x,y]\right) - \left[f\left(\mu_i^k x\right), f\left(\mu_i^k y\right)\right]\right\|_B \\ &\leq \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \alpha\left(\mu_i^k x, \mu_i^k y, 0, \cdots, 0\right) = 0\end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, $\mathcal{H}$ is a quadratic homomorphism. By $(A3)$, $\mathcal{H}$ is the unique fixed point of T in the set $\Delta = \{\mathcal{H} \in \Omega : d(f, \mathcal{H}) < \infty\}$, $\mathcal{H}$ is the unique function such that

$$\|f(x) - \mathcal{H}(x)\|_B \leq K\gamma(x)$$

for all $x \in \mathcal{A}$ and $K > 0$. Finally by $(A4)$, we obtain $d(f, \mathcal{H}) \leq \frac{1}{1-L} d(f, Tf)$ this implies $d(f, \mathcal{H}) \leq \frac{L^{1-i}}{1-L}$ which yields $\|f(x) - \mathcal{H}(x)\|_B \leq \frac{L^{1-i}}{1-L}\gamma(x)$.
This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.3).

Corollary 4.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x_1, y_1, \cdots, x_n, y_n)\|_B \leq \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \left(||x_i||_A^s + ||y_i||_A^s\right), & s \neq 4; \\ \lambda \left\{ \prod_{i=1}^{n} ||x_i||_A^s ||y_i||_A^s + \sum_{i=1}^{n} \left(||x_i||_A^{2ns} + ||y_i||_A^{2ns}\right)\right\}, & s \neq \frac{2}{n}; \end{cases} \tag{4.11}$$

for all $x_1, y_1, \cdots, x_n, y_n \in \mathcal{A}$ and

$$\|f[(x,y)] - [f(x), f(y)]\|_B \leq \begin{cases} \lambda, \\ \lambda \left\{||x||_A^s + ||y||_A^s\right\}, \\ \lambda \left\{||x||_A^s ||y||_A^s + \left(||x||_A^{2ns} + ||y||_A^{2ns}\right)\right\}, \end{cases} \tag{4.12}$$

for all $x, y \in \mathcal{A}$. Then there exists a unique quadratic homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_B \leq \begin{cases} \frac{\lambda}{|n^2 - 1|}, \\ \frac{2\lambda ||x||_A^{s/2}}{|n^2 - n^{s/2}|}, \\ \frac{2\lambda ||x||_A^{ns}}{|n^2 - n^{ns}|} \end{cases} \tag{4.13}$$

for all $x \in \mathcal{A}$.

Proof. Setting

$$\varphi(x_1,y_1,\cdots,x_n,y_n)=\begin{cases}\lambda,\\ \lambda\sum_{i=1}^{n}\left(||x_i||_A^s+||y_i||_A^s\right),\\ \lambda\left\{\prod_{i=1}^{n}||x_i||_A^s||y_i||_A^s+\sum_{i=1}^{n}\left(||x_i||_A^{2ns}+||y_i||_A^{2ns}\right)\right\}\end{cases}$$

for all $x_1,y_1,\cdots,x_n,y_n\in\mathcal{A}$. Now,

$$\frac{\varphi(\mu_i^k x_1,\mu_i^k y_1,\cdots,\mu_i^k x_n,\mu_i^k y_n)}{\mu_i^{2k}}$$

$$=\begin{cases}\lambda\mu_i^{-2k},\\ \lambda\mu_i^{(s-2)k}\sum_{i=1}^{n}(||x_i||_A^s+||y_i||_A^s),\\ \lambda\mu_i^{2(ns-1)k}\left\{\prod_{i=1}^{n}||x_i||_A^s||y_i||_A^s+\sum_{i=1}^{n}\left(||x_i||_A^{2ns}+||y_i||_A^{2ns}\right)\right\}\end{cases}=\begin{cases}\to 0 \text{ as } k\to\infty,\\ \to 0 \text{ as } k\to\infty,\\ \to 0 \text{ as } k\to\infty.\end{cases}$$

Thus, (4.1) is holds.

But we have $\gamma(x)=\varphi\left(\sqrt{\frac{x}{n}},\sqrt{\frac{x}{n}},\cdots,\sqrt{\frac{x}{n}}\right)$ has the property $\gamma(x)=L\cdot\frac{1}{\mu_i^2}\gamma(\mu_i x)$ for all $x\in\mathcal{A}$. Hence

$$\gamma(x)=\varphi\left(\sqrt{\frac{x}{n}},\sqrt{\frac{x}{n}},\cdots,\sqrt{\frac{x}{n}}\right)=\begin{cases}\lambda,\\ \lambda\frac{2n}{n^{s/2}}||x||_A^{s/2},\\ \lambda\left(\frac{1}{n^{ns}}+\frac{2n}{n^{ns}}\right)||x||_A^{ns}.\end{cases}$$

Now,

$$\frac{1}{\mu_i^2}\gamma(\mu_i x)=\begin{cases}\frac{\lambda}{\mu_i^2},\\ \frac{\lambda}{\mu_i^2}\frac{2n}{n^{s/2}}\left(||\mu_i x||_A^{s/2}\right),\\ \frac{\lambda}{\mu_i^2}\frac{1+2n}{n^{ns}}\left(||\mu_i x||_A^{ns}\right)\end{cases}=\begin{cases}\mu_i^{-2}\gamma(x),\\ \mu_i^{\frac{s-4}{2}}\gamma(x),\\ \mu_i^{ns-2}\gamma(x).\end{cases}$$

Hence the inequality (4.5) holds either, $L=n^{(s-4)/2}$ for $s<4$ if $i=0$ and $L=\frac{1}{n^{(s-4)/2}}$ for $s>4$ if $i=1$. Now from (4.6), we prove the following cases for condition (ii).

Case:1 $L=n^{(s-4)/2}$ for $s<4$ if $i=0$,

$$\|f(x)-\mathcal{H}(x)\|_B\le\frac{\left(n^{(s-4)/2}\right)^{1-0}}{1-n^{(s-4)/2}}\gamma(x)\le\frac{2n\lambda}{\left(n^2-n^{s/2}\right)}||x||_A^{s/2}.$$

Case:2 $L=n^{(4-s)/2}$ for $s>4$ if $i=1$,

$$\|f(x)-\mathcal{H}(x)\|_B\le\frac{\left(n^{(4-s)/2}\right)^{1-1}}{1-n^{(4-s)/2}}\gamma(x)\le\frac{2n\lambda}{\left(n^{s/2}-n^2\right)}||x||_A^{s/2}.$$

Similarly, the inequality (4.5) holds either, $L=n^{-2}$ for $s=0$ if $i=0$ and $L=\frac{1}{n^{-2}}$ for $s=0$ if $i=1$ for condition (i) and the inequality (4.5) holds either, $L=n^{ns-2}$ for $s<\frac{2}{n}$ if $i=0$ and $L=\frac{1}{n^{ns-2}}$ for $s>\frac{2}{n}$ if $i=1$ for condition (iii).

Hence the proof is complete □

Theorem 4.4. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there exist a function $\varphi : \mathcal{A}^{2n} \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \varphi(\mu_i^k x_1, \mu_i^k y_1, \mu_i^k x_2, \mu_i^k y_2, \cdots, \mu_i^k x_n, \mu_i^k y_n) = 0 \tag{4.14}$$

where $\mu_i = n$ if $i = 0$ and $\mu_i = \frac{1}{n}$ if $i = 1$ such that the functional inequality with

$$\|F(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)\|_B \le \varphi(x_1, y_1, x_2, y_2, \cdots, x_n, y_n) \tag{4.15}$$

for all $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \in \mathcal{A}$ and

$$\left\| f([x, y]) - \left[x^2, f(y)\right] - \left[f(x), y^2\right] \right\|_B \le \varphi(x, y, 0, \cdots, 0) \tag{4.16}$$

for all $x, y \in \mathcal{A}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \varphi\left(\sqrt{\frac{x}{n}}, \sqrt{\frac{x}{n}}, \cdots, \sqrt{\frac{x}{n}}\right), \tag{4.17}$$

has the property

$$\gamma(x) = L \frac{1}{\mu_i^2} \gamma(\mu_i x). \tag{4.18}$$

for all $x \in \mathcal{A}$. Then there exists a unique quadratic derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (1.3) and

$$\| f(x) - \mathcal{D}(x) \|_B \le \frac{L^{1-i}}{1 - L} \gamma(x) \tag{4.19}$$

for all $x \in \mathcal{A}$.

Proof. By the same reasoning as that in the proof of Theorem 4.3, there exists a unique quadratic mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying (4.19). The mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is given by $\mathcal{D}(x) = \lim_{k\to\infty} \frac{f\left(\mu_i^k x\right)}{\mu_i^{2k}}$ for all $x \in \mathcal{A}$. It follows from (4.15) that

$$\begin{aligned} \left\| \mathcal{D}\left[(x, y)\right] - \left[x^2, \mathcal{D}(y)\right] - \left[\mathcal{D}(x), y^2\right] \right\|_B &\le \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \left\| f\left(\mu_i^{2k} x, y\right) - (\mu_i x_1)^2 f\left(\mu_i^k x_3\right) - f\left(\mu_i^k x_1\right)\left(\mu_i^k x_3\right)^2 \right\|_B \\ &\le \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \varphi\left(\mu_i^k x, \mu_i^k y, 0, \cdots, 0\right) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is a quadratic derivation satisfying (4.19). The rest of the proof is similar to that of Theorem 4.3. □

The following corollary is an immediate consequence of Theorem 4.4 concerning the stability of (1.3).

Corollary 4.4. *Let $F : \mathcal{A} \to \mathcal{A}$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x_1, y_1, \cdots, x_n, y_n)\|_B \le \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \left(||x_i||_A^s + ||y_i||_A^s\right), & s \ne 4; \\ \lambda \left\{ \prod_{i=1}^{n} ||x_i||_A^s ||y_i||_A^s + \sum_{i=1}^{n} \left(||x_i||_A^{2ns} + ||y_i||_A^{2ns}\right) \right\}, & s \ne \frac{2}{n}; \end{cases} \tag{4.20}$$

for all $x_1, y_1, \cdots, x_n, y_n \in \mathcal{A}$ and

$$\left\| f([x, y]) - \left[x^2, f(y)\right] - \left[f(x), y^2\right] \right\|_B \le \begin{cases} \lambda, \\ \lambda \left\{ ||x||_A^s + ||y||_A^s \right\}, \\ \lambda \left\{ ||x||_A^s ||y||_A^s + \left(||x||_A^{2ns} + ||y||_A^{2ns}\right) \right\}, \end{cases} \tag{4.21}$$

for all $x, y \in \mathcal{A}$. Then there exists a unique quadratic derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \mathcal{D}(x)\|_B \leq \begin{cases} \dfrac{\lambda}{|n^2-1|}, \\ \dfrac{2\lambda||x||_A^{s/2}}{|n^2-n^{s/2}|}, \\ \dfrac{2\lambda||x||_A^{ns}}{|n^2-n^{ns}|} \end{cases} \tag{4.22}$$

for all $x \in \mathcal{A}$.

5 Application of the Functional Equation (1.3)

Consider the functional equation (1.3), that is

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} f(y_i)\right) = f\left(\sum_{i=1}^{n} x_i y_i\right) + \sum_{1\leq i<j\leq n} f\left(x_i y_j - x_j y_i\right).$$

Since $f(x) = x^2$ is the solution of the functional equation, then the above equation can be written as

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) = \left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{1\leq i<j\leq n} \left(x_i y_j - x_j y_i\right)^2 \tag{5.23}$$

for $x_i, y_i, i = 1, 2, \cdots, n$, real or complex numbers.

Attributed to Joseph Louis Lagrange [17], with several fields of mathematics and mechanics, a special case of (5.23) is found in Fibonacci's Book of Squares (in the original Latin words, Liber Quadratorum):

$$\left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right) = (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2. \tag{5.24}$$

For integer value of the variables, this means that "the product of sums of squares is again a sum of squares".

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