Abstract. This article introduces $\theta_f$-approximations of sets in fuzzy proximal relator space where $\theta \in [0, 1)$. $\theta_f$-approximation provides a more sensitive approach for the upper approximations of subsets or subimages. $\theta_f$-approximation of a subimage are given with an example in digital images. Furthermore, $\theta_f$-approximately groupoid and semigroup in fuzzy proximal relator space are introduced.

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Keywords: Proximity space, relator space, descriptive approximation, approximately semigroup.

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1. Introduction

Efremović discovered the proximity spaces in 1951 [2]. He defined proximity space using with proximity relation for proximity of arbitrary subsets of a set. In [10], one can find a list of publications on proximity spaces. A proximity measure is a measure of the closeness or nearness between two nonempty sets.

A relator is a set of binary relations on a nonempty set $X$ that is denoted by $R$. A relator space is defined as the pair $(X, R)$. In 2016, Peters introduced the concept of proximal relator space $(X, \mathcal{R}_\delta)$ where $\mathcal{R}_\delta$ is a family of proximity relations on $X$ [14].

Zadeh defined fuzzy sets in 1965, which he interpreted as a generalization of set. A fuzzy set $A$ in a universe $X$ is a mapping $A : X \rightarrow [0, 1]$ [23]. For some applications of fuzzy sets please see [16, 17, 19]. Fuzzy similarity measure between fuzzy sets are given in [22]. Fuzzy similarity measure between sets using with fuzzy proximity relation $\mu_R$ and fuzzy proximal relator space $(X, \mu_R)$ are introduced in [11]. Studies in the field of algebraic topology were also discussed with a different perspective on these issues, and semitopological $\delta$-groups were published in 2023 [7].

Fuzzy similarity measures and fuzzy proximity relations are useful tools for applications in the applied sciences such as digital image processing and computer vision. A digital image endowed with fuzzy proximity

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relation $\mu_R$ is fuzzy proximal relator space. Therefore one can working on pixels in digital images to obtain classifications or approximations.

Let $A \subseteq X$. A descriptively upper approximation of $A$ is defined with

$$\Phi^*A = \{x \in X \mid x \delta_{\Phi} A\}.$$ 

It means, descriptively upper approximation of $A$ consists of all elements in $X$ that only have exactly the same properties as elements in $A$. But sometimes more sensitive calculations may be needed.

For more sensitive approach to compute upper approximations of subsets, we can also consider elements with somewhat similar properties, even if they do not have the same properties. To do this, the concept of fuzzy set is one of the most effective mathematical tool. Therefore, $\theta_f$-approximations of sets in fuzzy proximal relator space are developed. Main advantages of this study is that it effectively uses the concepts of fuzzy sets, proximity relations and upper approximation of sets together.

In section 2, definitions of Efremović proximity, set description, descriptively near sets, descriptively upper approximations of sets in fuzzy proximal relator space are introduced, where $\theta \in [0,1)$. $\theta_f$-approximation provides a more sensitive approach for the upper approximations of subsets or subimages. $\theta_f$-approximation of a subimage are given with an example in digital images. Furthermore, $\theta_f$-approximately groupoid and semigroup in fuzzy proximal relator space are introduced.

2. Preliminaries

**Definition 2.1.** [2, 3] Let $X$ be a nonempty set and $\delta$ be a relation on $P(X)$. $\delta$ is called an Efremović proximity that satisfy following axioms:

1. $A \delta B$ implies $B \delta A$.
2. $A \delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$.
3. $A \cap B \neq \emptyset$ implies $A \delta B$.
4. $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$.
5. $\{x\} \delta \{y\}$ iff $x = y$.
6. $A \delta B$ implies $\exists E \subseteq X$ such that $A \delta E$ and $E^c \delta B$.

for all $A, B, C \in P(X)$ and all $x, y \in X$. Efremović proximity relation is denoted by $\delta_E$.

**Definition 2.2.** [9] Let $X$ be a nonempty set and $\delta$ be a relation on $P(X)$. $\delta$ is called a Lodato proximity that satisfy the axioms $(A_1)-(A_3)$ and

1. $A \delta B$ and $\{b\} \delta C$ ($\forall b \in B$) implies $A \delta C$ for all $A, B, C \in P(X)$. Lodato proximity relation is denoted by $\delta_L$.

Let $X$ be a nonempty set and $\mathcal{R}$ be a set of relations on $X$. $\mathcal{R}$ and $(X, \mathcal{R})$ is called a relator and a relator space, respectively [20]. Let $\mathcal{R}_{\delta}$ be a family of proximity relations on $X$. Then $(X, \mathcal{R}_{\delta})$ is a proximal relator space. As in [14], $\mathcal{R}_{\delta}$ contains proximity relations such as basic proximity $\delta_E$ [18], Efremović proximity $\delta_E$ [2, 3], Lodato proximity $\delta_L$ [9], Wallman proximity $\delta_W$ [21], descriptive proximity $\delta_{\Phi}$ [12, 15].

In a discrete space, a non-abstract point has a location and features. Features can be measured using probe functions [8]. Let $X$ be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_E})$.

In this space, a function $\Phi : X \rightarrow \mathbb{R}^n$, $\Phi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ is an object description represents a feature vector of $x \in X$ where each $\varphi_i : X \rightarrow \mathbb{R}$ is a probe function ($1 \leq i \leq n$) that describes feature of a non-abstract point such as pixel in a digital image.

Throughout this work, nonempty set of non-abstract points $X$ was considered. Efremović proximity $\delta_E$ [3] and descriptive proximity $\delta_{\Phi}$ in defining a descriptive proximal relator space $(X, \mathcal{R}_{\delta_{\Phi}})$ were considered. Also, instead of the notions proximal relator space and fuzzy proximal relator space, the terms $PR$-space and $FPR$-space were used briefly, respectively.
Definition 2.3. [10] Let \( \Phi \) be an object description and \( A \subseteq X \). Then the set description of \( A \) is defined as
\[
Q(A) = \{ \Phi(a) \mid a \in A \}.
\]

Definition 2.4. [10, 13] Let \( A, B \subseteq X \). Then the descriptive (set) intersection of \( A \) and \( B \) is defined as
\[
A \cap_{\Phi} B = \{ x \in A \cup B \mid \Phi(x) \in Q(A) \text{ and } \Phi(x) \in Q(B) \}.
\]

Definition 2.5. [12] Let \( \delta_{\Phi} \in \mathcal{R}_{\Phi} \) and \( A, B \subseteq X \). If \( Q(A) \cap Q(B) \neq \emptyset \), then \( A \) is called a descriptively near \( B \) and denoted by \( A \delta_{\Phi} B \). If \( Q(A) \cap Q(B) = \emptyset \), then \( A \delta_{\Phi} B \) reads \( A \) is descriptively far from \( B \).

Definition 2.6. [22] Let \( X \) be an universal set and \( \mathcal{F}(X) \) be a class of all fuzzy sets of \( X \). A function \( \mu : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0, 1] \) is called a fuzzy similarity measure if the following axioms satisfy:
\[
(\mu_1) \mu(A, \emptyset) = 0 \text{ and } \mu(A, A) = 1,
(\mu_2) \mu(A, B) = \mu(B, A),
(\mu_3) A \subseteq B \subseteq C \text{ implies } \mu(A, B) \geq \mu(A, C) \text{ and } \mu(B, C) \geq \mu(A, C)
\]
for all \( A, B, C \in \mathcal{F}(X) \).

Definition 2.7. [11] Let \((X, \mathcal{R})\) be a PR-space,
\[
\mu_{\mathcal{R}} : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, 1]
\]
be a fuzzy relation and \( A, B \subseteq X \). Then \( \mu_{\mathcal{R}} \) is called a fuzzy proximity relation if it satisfies the following axioms:
\[
(\mu_{\mathcal{R}})_1 \mu_{\mathcal{R}}(A, \emptyset) = 0,
(\mu_{\mathcal{R}})_2 \mu_{\mathcal{R}}(A, B) = \mu_{\mathcal{R}}(B, A),
(\mu_{\mathcal{R}})_3 \mu_{\mathcal{R}}(A, B) \neq 0 \text{ implies } A \text{ is fuzzy proximal to } B,
(\mu_{\mathcal{R}})_4 \mu_{\mathcal{R}}(A, B \cup C) \neq 0 \text{ implies } \mu_{\mathcal{R}}(A, B) \neq 0 \text{ or } \mu_{\mathcal{R}}(A, C) \neq 0
\]
for all \( A, B, C \in \mathcal{P}(X) \).

The set of all fuzzy proximity relations on \( \mathcal{P}(X) \) is denoted by \( \mathcal{P}_{\mu_{\mathcal{R}}}(X) \). Therefore \( \mu_{\mathcal{R}}(A, B) \) is called a fuzzy proximity measure of \( A \) with \( B \).

If \( \mu_{\mathcal{R}}(A, B) > 0 \), then \( A \) is fuzzy proximal to \( B \). Also, if \( \mu_{\mathcal{R}}(A, B) > \theta \), then \( A \) is \( \theta \)-fuzzy proximal to \( B \) for \( \theta \in (0, 1) \).

Definition 2.8. [11] Let \((X, \mathcal{R})\) be a PR-space and \( \mu_{\mathcal{R}} \) be a fuzzy proximity relation. Then \((X, \mathcal{R}, \mu_{\mathcal{R}})\) is called a FPR-space and shortly denoted by \((X, \mu_{\mathcal{R}})\).

3. \( \theta_{\mathcal{R}} \)-Approximations and \( \theta_{\mathcal{R}} \)-Approximately Semigroups

Definition 3.1. Let \((X, \mu_{\mathcal{R}})\) be a FPR-space and \( A \subseteq X \). A \( \theta_{\mathcal{R}} \)-approximation of \( A \) is determined with
\[
A^{\theta}_{\mu_{\mathcal{R}}} = \bigcup_{\mu_{\mathcal{R}}(A, B) > \theta} B,
\]
where \( B \in \mathcal{P}(X) \) and \( \theta \in [0, 1) \).

For clarify the mechanism of \( \theta_{\mathcal{R}} \)-approximation please see Example 3.3.

Example 3.2. Let \( X \) be a digital image and \( x, y \) be pixels of \( X \). Probe functions \( \varphi(x) = (R_x, G_x, B_x) \) and \( \varphi_y = (R_y, G_y, B_y) \) are represent the RGB codes of pixels \( x, y \). Let
\[
\mu_{\mathcal{R}} : X \times X \rightarrow [0, 1]
\]

\[
(x, y) \mapsto \mu_{\mathcal{R}}(x, y) = \frac{|765 - D_{x,y}|}{765}
\]
be a fuzzy relation where
\[ D_{x,y} = \sqrt{2 \Delta R^2 + 4 \Delta G^2 + 3 \Delta B^2} \]
is a weighted Euclidean distance of pixels with respect to RGB such that \( \Delta R = R_x - R_y, \Delta G = G_x - G_y \) and \( \Delta B = B_x - B_y \). In the definition of \( \mu_R \), 765 is the maximum value of \( D_{x,y} \).

Furthermore, fuzzy relationship between \( x \) and \( y \cup z \) means that
\[ \mu_R(x, y \cup z) = \frac{|765 - \min\{D_{x,y}, D_{x,z}\}|}{765} \]
for all \( x, y, z \in X \).

Now let’s show that \( \mu_R \) is a fuzzy proximity relation.

(\( \mu_R \)) \( _1 \) Since there is no similarity between \( x \in X \) and \( \emptyset \), it is clear that \( \mu_R(x, \emptyset) = 0 \).

(\( \mu_R \)) \( _2 \) \( \mu_R(x, y) = \mu_R(y, x) \) by \( D_{x,y} = D_{y,x} \) for all \( x, y \in X \).

(\( \mu_R \)) \( _3 \) Obviously \( \mu_R(x, y) \neq 0 \) implies \( A \) is fuzzy proximal to \( B \).

(\( \mu_R \)) \( _4 \) Let \( \mu_R(x, y) = 0 \) and \( \mu_R(x, z) = 0 \) for all \( x, y, z \in X \). Then \( \mu_R(x, y) = \frac{|765 - D_{x,y}|}{765} = 0 \), that is, \( D_{x,y} = 765 \). Similarly \( D_{x,z} = 765 \). Hence \( D_{x,y} = D_{x,z} = 765 \) and \( \min\{D_{x,y}, D_{x,z}\} = 765 \). Thus \( \mu_R(x, y \cup z) = \frac{|765 - \min\{D_{x,y}, D_{x,z}\}|}{765} = 0 \). Therefore \( \mu_R(x, y \cup z) \neq 0 \) implies \( \mu_R(x, y) \neq 0 \) or \( \mu_R(x, z) \neq 0 \) for all \( x, y, z \in X \).

Consequently, \( \mu_R \) is a fuzzy proximity relation from Definition 2.7.

**Example 3.3.** Let \( X \) be a digital image consists of 16 pixels as in Fig. 1. Also, digital image \( X \) endowed with fuzzy proximity relation \( \mu_R \) from Example 3.2 is a F P R-space by Definition 2.8.

![Figure 1: Digital image X](image)

In digital image \( X \), a pixel \( x_{ij} \) is an element at position \( (i, j) \) (row and column). Table 1 lists the RGB codes for each pixel.

**Table 1.** RGB codes for each pixel in \( X \).

<table>
<thead>
<tr>
<th></th>
<th>( x_{11} )</th>
<th>( x_{12} )</th>
<th>( x_{13} )</th>
<th>( x_{14} )</th>
<th>( x_{21} )</th>
<th>( x_{22} )</th>
<th>( x_{23} )</th>
<th>( x_{24} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>117</td>
<td>91</td>
<td>91</td>
<td>180</td>
<td>180</td>
<td>110</td>
<td>132</td>
<td>117</td>
</tr>
<tr>
<td>Green</td>
<td>213</td>
<td>165</td>
<td>149</td>
<td>227</td>
<td>227</td>
<td>161</td>
<td>188</td>
<td>213</td>
</tr>
<tr>
<td>Blue</td>
<td>215</td>
<td>227</td>
<td>227</td>
<td>228</td>
<td>228</td>
<td>230</td>
<td>234</td>
<td>215</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( x_{31} )</th>
<th>( x_{32} )</th>
<th>( x_{33} )</th>
<th>( x_{34} )</th>
<th>( x_{41} )</th>
<th>( x_{42} )</th>
<th>( x_{43} )</th>
<th>( x_{44} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>183</td>
<td>91</td>
<td>226</td>
<td>180</td>
<td>117</td>
<td>226</td>
<td>117</td>
<td>226</td>
</tr>
<tr>
<td>Green</td>
<td>253</td>
<td>165</td>
<td>226</td>
<td>227</td>
<td>213</td>
<td>226</td>
<td>213</td>
<td>226</td>
</tr>
<tr>
<td>Blue</td>
<td>233</td>
<td>227</td>
<td>226</td>
<td>228</td>
<td>215</td>
<td>226</td>
<td>215</td>
<td>226</td>
</tr>
</tbody>
</table>

From Example 3.2, values of fuzzy proximity relation \( \mu_R \) are given in Table 2.
Lemma 3.4. Let \((X, \mu_R)\) be a FPR-space, \(A \subseteq X\) and \(\theta \in [0,1)\). Then the following statements hold:
(i) \(\emptyset^\theta_{\mu_R} = \emptyset\),
(ii) \(A \subseteq A^\theta_{\mu_R}\),
(iii) \(X^\theta_{\mu_R} = X\).
Proof. It is straightforward.

Theorem 3.5. Let \((X, \mu_R)\) be a FPR-space, \(A, B \subseteq X\) and \(\theta \in [0, 1)\). Then the following statements hold:

(i) If \(A \subseteq B\), then \(B^\theta_{\mu_R} \subseteq A^\theta_{\mu_R}\).

(ii) \((A^\theta_{\mu_R})_{\mu_R} = A^\theta_{\mu_R}\).

(iii) \(A^\theta_{\mu_R} \cap B^\theta_{\mu_R} \subseteq (A \cap B)^\theta_{\mu_R}\).

(iv) \((A \cup B)^\theta_{\mu_R} \subseteq A^\theta_{\mu_R} \cup B^\theta_{\mu_R}\).

Proof. (i) Let \(A \subseteq B\) and \(x \in B^\theta_{\mu_R}\) where \(\theta \in [0, 1)\). Then \(\mu_R(B, x) > \theta\) and hence \(\mu_R(A, x) > \theta\) since \(A \subseteq B\). Thus \(x \in A^\theta_{\mu_R}\). Therefore \(B^\theta_{\mu_R} \subseteq A^\theta_{\mu_R}\).

(ii) It is clear that \((A^\theta_{\mu_R})_{\mu_R} \subseteq A^\theta_{\mu_R}\) from (i). Also, \(A^\theta_{\mu_R} \subseteq (A^\theta_{\mu_R})_{\mu_R}\) by Lemma 3.4 (ii) and so \((A^\theta_{\mu_R})_{\mu_R} = A^\theta_{\mu_R}\).

(iii) Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), from (i) \(A^\theta_{\mu_R} \subseteq (A \cap B)^\theta_{\mu_R}\) and \(B^\theta_{\mu_R} \subseteq (A \cap B)^\theta_{\mu_R}\). Thus \(A^\theta_{\mu_R} \cap B^\theta_{\mu_R} \subseteq (A \cap B)^\theta_{\mu_R}\).

(iv) Because of \(A \subseteq A \cup B\) and \(B \subseteq A \cup B\), \((A \cup B)^\theta_{\mu_R} \subseteq A^\theta_{\mu_R}\) and \((A \cup B)^\theta_{\mu_R} \subseteq B^\theta_{\mu_R}\) from (i). Therefore \((A \cup B)^\theta_{\mu_R} \subseteq A^\theta_{\mu_R} \cup B^\theta_{\mu_R}\).

Theorem 3.6. Let \((X, \mu_R)\) be a FPR-space, \(A_i \subseteq X\) \((i = 1, 2, \ldots, n), n \in \mathbb{N}\) and \(\theta \in [0, 1)\). Then the following statements hold:

(i) \(\bigcap_{i} (A_i)^\theta_{\mu_R} \subseteq \left( \bigcap_{i} A_i \right)^\theta_{\mu_R}\).

(ii) \(\left( \bigcup_{i} A_i \right)^\theta_{\mu_R} \subseteq \bigcup_{i} (A_i)^\theta_{\mu_R}\).

Theorem 3.7. Let \((X, \mu_R)\) be a FPR-space, \(A \subseteq X\) and \(\theta_i \in [0, 1)\) \((i = 1, 2, \ldots, n), n \in \mathbb{N}\). Then

(i) If \(\theta_1 > \theta_2\), then \(A^\theta_{\mu_R} \subseteq A^\theta_{\mu_R}\).

(ii) If \(\theta_1 > \theta_2 > \cdots > \theta_n\), then \(A^\theta_{\mu_R} \subseteq A^\theta_{\mu_R} \subseteq \cdots \subseteq A^\theta_{\mu_R}\).

Proof. (i) Let \(\theta_1 > \theta_2\) and \(x \in A^\theta_{\mu_R}\). Then \(\mu_R(A, x) > \theta_1\) and so \(\mu_R(A, x) > \theta_2\) since \(\theta_1 > \theta_2\). Hence \(x \in A^\theta_{\mu_R}\). Consequently, \(A^\theta_{\mu_R} \subseteq A^\theta_{\mu_R}\).

(ii) It is easily obtained from (i).

Theorem 3.8. Let \((X, \mu_R)\) be a FPR-space, \(A \subseteq X\), \(\theta_i \in [0, 1), n \in \mathbb{N}\) and \(\bigwedge_{i} \theta_i = \alpha\), \(\bigvee_{i} \theta_i = \beta\). Then

(i) \(\bigcup_{i} A^\theta_{\mu_R} = A^\alpha_{\mu_R}\).

(ii) \(\bigcap_{i} A^\theta_{\mu_R} = A^\beta_{\mu_R}\).

Proof. (i) Let \(x \in \bigcup_{i} A^\theta_{\mu_R}\). Then \(x \in A^\theta_{\mu_R}\) and so \(\mu_R(A, x) > \theta_i\) for at least \(i\). Hence \(\mu_R(A, x) > \alpha\) from \(\bigwedge_{i} \theta_i = \alpha\). Thus \(x \in A^\alpha_{\mu_R}\). Therefore \(\bigcup_{i} A^\theta_{\mu_R} \subseteq A^\alpha_{\mu_R}\). Similarly, we can show that \(A^\alpha_{\mu_R} \subseteq \bigcup_{i} A^\theta_{\mu_R}\). As a result, \(\bigcup_{i} A^\theta_{\mu_R} = A^\alpha_{\mu_R}\) for all \(i\).

(ii) Let \(x \in \bigcap_{i} A^\theta_{\mu_R}\). Then \(x \in A^\theta_{\mu_R}\) and so \(\mu_R(A, x) > \theta_i\) for all \(i\). Hence \(\mu_R(A, x) > \beta\) from \(\bigvee_{i} \theta_i = \beta\). Thus \(x \in A^\beta_{\mu_R}\). Therefore \(\bigcap_{i} A^\theta_{\mu_R} \subseteq A^\beta_{\mu_R}\). Similarly, we can show that \(A^\beta_{\mu_R} \subseteq \bigcap_{i} A^\theta_{\mu_R}\). Consequently, \(\bigcap_{i} A^\theta_{\mu_R} = A^\beta_{\mu_R}\) for all \(i\).

Definition 3.9. Let \((X, \mu_R)\) be a FPR-space and let \(\cdot\) be a binary operation defined on \(X\). \(G \subseteq X\) is called a \(\theta_f\)-approximately groupoid in FPR-space if \(x \cdot y \in G^\theta_{\mu_R}\) for all \(x, y \in G\).
\(\theta_f\)-Approximations via fuzzy proximity relations: Semigroups in digital images

Let we consider \(G\) is a \(\theta_f\)-approximately groupoid with the operation \(\cdot\) in \((X, \mu_R)\), \(g \in G\) and \(A, B \subseteq G\). The subsets \(g \cdot A, A \cdot g, A \cdot B \subseteq C^0_{\mu_R} \subseteq X\) are described as follows:

\[
g \cdot A = gA = \{ga | a \in A\}, \\
A \cdot g = Ag = \{ag | a \in A\}, \\
A \cdot B = AB = \{ab | a \in A, b \in B\}.
\]

**Definition 3.10.** Let \((X, \mu_R)\) be a FPR-space, \(\cdot\) be a binary operation on \(X\) and \(S \subseteq X\). \(S\) is named a \(\theta_f\)-approximately semigroup in FPR-space if the conditions mentioned below are obtained:

1. \(x \cdot y \in S^\theta_{\mu_R}\),
2. \((x \cdot y) \cdot z = x \cdot (y \cdot z)\) property satisfy on \(S^\theta_{\mu_R}\) for all \(x, y, z \in S\).

If \(\theta_f\)-approximately semigroup has approximately identity element \(e_\theta \in S^\theta_{\mu_R}\) such that \(x \cdot e_\theta = e_\theta \cdot x = x\) for all \(x \in S\), then \(S\) is called a \(\theta_f\)-approximately monoid in FPR-space. If \(x \cdot y = y \cdot x\) property holds in \(S^\theta_{\mu_R}\) for all \(x, y \in S\), then \(S\) is commutative \(\theta_f\)-approximately semigroup in FPR-space.

**Example 3.11.** Assume \(X\) is a 16 pixel digital image, as shown in Fig. 1 and \(S = \{x_{13}, x_{23}, x_{32}\}\) be a subimage of \(X\). From Example 3.3, \(\theta_f\)-approximation of \(S\) is

\[
S^\theta_{\mu_R} = \{x_{13}, x_{23}, x_{32}, x_{12}, x_{22}\}
\]

where \(\theta = 0.92\).

Let

\[
\cdot: X \times X \rightarrow X, \\
(x_{ij}, x_{kl}) \mapsto x_{ij} \cdot x_{kl} = x_{pr}
\]

be a binary operation on \(X\) such that \(p = \min \{i, k\}\) and \(r = \min \{j, l\}\).

By Definition 3.10, since

1. \(x_{ij} \cdot x_{kl} \in S^\theta_{\mu_R}\),
2. \((x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})\) property satisfy on \(S^\theta_{\mu_R}\) for all \(x_{ij}, x_{kl}, x_{mn} \in S\) are satisfied, \(S\) is indeed a \(\theta_f\)-approximately semigroup in FPR-space \((X, \mu_R)\) with \(\cdot\).

Also, since \(x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}\) for all \(x_{ij}, x_{kl} \in S\) property satisfies in \(S^\theta_{\mu_R}\), \(S\) is a commutative \(\theta_f\)-approximately semigroup.

**Definition 3.12.** Let \((X, \mu_R)\) be a FPR-space, \(S \subseteq X\) be a \(\theta_f\)-approximately semigroup and \(T \subseteq S\) \((T \neq \emptyset)\). \(T\) is called a \(\theta_f\)-approximately subgroup if \(T\) is a \(\theta_f\)-approximately semigroup with the operation in \(S\).

**Theorem 3.13.** Let \(S\) be a \(\theta_f\)-approximately semigroup and \(T \subseteq S\) \((T \neq \emptyset)\). If \(T^\theta_{\mu_R}\) is a \(\theta_f\)-approximately groupoid and \(T^\theta_{\mu_R} \subseteq S^\theta_{\mu_R}\), then \(T\) is a \(\theta_f\)-approximately subsemigroup of \(S\).

**Proof.** Since \(T^\theta_{\mu_R}\) is a \(\theta_f\)-approximately groupoid, thus \(x \cdot y \in T^\theta_{\mu_R}\) for all \(x, y \in T\). Furthermore, \((x \cdot y) \cdot z = x \cdot (y \cdot z)\) property satisfies on \(T^\theta_{\mu_R}\) for all \(x, y, z \in T\), since \(S\) is a \(\theta_f\)-approximately semigroup and \(T^\theta_{\mu_R} \subseteq S^\theta_{\mu_R}\). Consequently, \(T\) is a \(\theta_f\)-approximately subsemigroup of \(S\).

**Definition 3.14.** Let \((X, \mu_R)\) be a FPR-space, \(S \subseteq X\) be a \(\theta_f\)-approximately semigroup and \(I \subseteq S\).

1. \(I\) is called a \(\theta_f\)-approximately left ideal of \(S\) if \(I^\theta_{\mu_R}\) is a left ideal of \(S\), i.e., \(S(I^\theta_{\mu_R}) \subseteq I^\theta_{\mu_R}\).
2. \(I\) is called a \(\theta_f\)-approximately right ideal of \(S\) if \(I^\theta_{\mu_R}\) is a right ideal of \(S\), i.e., \((I^\theta_{\mu_R})S \subseteq I^\theta_{\mu_R}\).
3. \(I\) is called a \(\theta_f\)-approximately bi-ideal of \(S\) if \(I^\theta_{\mu_R}\) is a bi-ideal of \(S\), i.e., \((I^\theta_{\mu_R})S(I^\theta_{\mu_R}) \subseteq I^\theta_{\mu_R}\).

**Example 3.15.** In Example 3.11, let we use \(\theta_f\)-approximately semigroup \(S = \{x_{13}, x_{23}, x_{32}\}\). From Definition 3.14, obviously \(S \subseteq S\) is a \(\theta_f\)-approximately left ideal, \(\theta_f\)-approximately right ideal and also \(\theta_f\)-approximately bi-ideal of \(S\).
**Theorem 3.16.** Let \( (X, \mu_R) \) be a \( \text{FPR-space} \) and \( S \subseteq X \). If \( S \) is a semigroup in \( X \), then \( S \) is a \( \theta_f \)-approximately semigroup in \( \text{FPR-space} \).

**Proof.** Assume that \( S \subseteq X \) be a semigroup. Using Lemma 3.4 (ii), \( S \subseteq S^\theta_{\mu_R} \) is obtained. Hence \( x \cdot y \in S^\theta_{\mu_R} \) and \((x \cdot y) \cdot z = x \cdot (y \cdot z) \) condition is also accurate in \( S^\theta_{\mu_R} \) for all \( x, y, z \in S \). After that \( S \) is a \( \theta_f \)-approximately semigroup in \( \text{FPR-space} \).

The Theorem 3.16 shows that \( \theta_f \)-approximately semigroup is a generalization of a semigroup.

**Theorem 3.17.** Let \( (X, \mu_R) \) be a \( \text{FPR-space} \) and \( S \subseteq X \). If \( I \) is a left (right) ideal of \( \theta_f \)-approximately semigroup \( S \) and \( (S^\theta_{\mu_R}) (I^\theta_{\mu_R}) \subseteq (I^\theta_{\mu_R}) (S^\theta_{\mu_R}) \subseteq I^\theta_{\mu_R} \), then \( I \) is a \( \theta_f \)-approximately left (right) ideal of \( S \).

**Proof.** Let we consider \( I \) be a left ideal of \( \theta_f \)-approximately semigroup \( S \), that is, \( SI \subseteq I \). We know that \( S \subseteq S^\theta_{\mu_R} \). Hence, from the hypothesis \( (S^\theta_{\mu_R}) (I^\theta_{\mu_R}) \subseteq I^\theta_{\mu_R} \),

\[
S (I^\theta_{\mu_R}) \subseteq (S^\theta_{\mu_R}) (I^\theta_{\mu_R}) \subseteq I^\theta_{\mu_R}.
\]

As a results, \( I^\theta_{\mu_R} \) is a left ideal of \( S \) and so \( I \) is a \( \theta_f \)-approximately left ideal of \( S \). Also, It is obviously if \( I \) is a right ideal of \( \theta_f \)-approximately semigroup \( S \) and \( (I^\theta_{\mu_R}) (S^\theta_{\mu_R}) \subseteq I^\theta_{\mu_R} \), \( I \) is a \( \theta_f \)-approximately right ideal of \( S \).

**4. Conclusions**

This work proposed \( \theta_f \)-approximations of sets in fuzzy proximal relator space to provide more sensitive approach for approximations or clustering. From the examples and results, it was verified that \( \theta_f \)-approximation is able to classify the pixels in digital images more precisely according to the selected \( \theta \in [0, 1] \). Other results about \( \theta_f \)-approximately algebraic structures provides a theoretical basis for further studies. Future studies should investigate the performance of this theory with experimental studies in any applied fields.

**References**


\(\theta_f\)-Approximations via fuzzy proximity relations: Semigroups in digital images


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