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Intuitionistic fuzzy unitary operator on intuitionistic fuzzy Hilbert space

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Abstract

In this paper, we define Intuitionistic fuzzy unitary operator (IFU-operator) on an intuitionistic fuzzy Hilbert space (IFH-space). An operator $\mathfrak{U} \in IFB(\mathbb{H})$ is intuitionistic fuzzy unitary operator if $\mathfrak{U}\mathfrak{U}^* = I = \mathfrak{U}^*\mathfrak{U}$ i.e. it is an isomorphism of $\mathbb{H}$ onto itself. By virtue of this definition, a few theorems on IFU-operator are introduced and some of its properties are discussed.

Keywords

Intuitionistic fuzzy adjoint operator (IFA-operator), intuitionistic fuzzy Hilbert space (IFH-space), intuitionistic fuzzy normal operator (IFN-operator), intuitionistic fuzzy self-adjoint operator (IFSA-operator), intuitionistic fuzzy unitary operator (IFU-operator).

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

Initially, Atanossov [9] in 1986, introduced the concept of intuitionistic fuzzy set. In 2004, Park [6] defined the notion of intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ with the help of continuous t-norm $*$ and continuous t-conorm $\diamond$ as a generalization of fuzzy metric space. Later in 2005 Saadati and Park [10], using the idea of intuitionistic fuzzy metric spaces, introduced intuitionistic fuzzy normed spaces. In 2009, Goudarzi et al. [5] presented the new notion of intuitionistic fuzzy normed spaces and introduced intuitionistic fuzzy inner product space (IFIP-space) with the help of continuous t-representable $(\mathcal{T})$ as a 3-tuple $(V, \mathbb{F}_{\mu,\nu}, \mathcal{T})$. Majumdar and Samanta [12] given the definition of IFIP (μ, μ^*) in a linear space, an IFIP-space (V, μ, μ^*) and some of their properties. In 2018, Radharamani et al. [1, 2] introduced the definition and properties of IFH-space and also the concept of IFA and IFSA operators in IFH-space. $\mathcal{S}^* \in IFB(V)$ is an IF-adjoint of an operator $\mathcal{S} \in IFB(V)$, if $\langle \mathcal{S}x, y\rangle = \langle x, \mathcal{S}^*y\rangle, \forall x, y \in V$, where IFB$(V)$ means the set of all Intuitionistic Fuzzy Bounded (continuous) linear operators on V. Also, if $\mathcal{S} = \mathcal{S}^*$, then $\mathcal{S}$ is an IFSA-operator. Radharamani et al. [3] defined the concept of Intuitionistic Fuzzy Normal operator on IFH-space and their properties. An operator $\mathcal{S} \in IFB(V)$ is called an IFN-operator iff $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$.

In this paper, we introduce definition of Intuitionistic Fuzzy Unitary operator (IFU-operator) on $\mathbb{H}$, if $\mathfrak{U}\mathfrak{U}^* = I = \mathfrak{U}^*\mathfrak{U}$. Here we give some properties of IFU-operator in IFH-space like, $\mathfrak{U}$ is an IFU-operator $\Leftrightarrow \mathcal{P}_{\mu,\nu}(\mathfrak{U}x, t) = \mathcal{P}_{\mu,\nu}(x, t), \forall x \in \mathbb{H}$. Sum of two IFU-operators is IFU iff it is surjective and $\mathrm{Re}(\mathfrak{U}_1 x, \mathfrak{U}_2 x) = -\frac{1}{2}, x \in \mathbb{H}$ with $\mathcal{P}_{\mu,\nu}(\mathfrak{U}, t) = 1$. $\mathfrak{U} \in IFB(\mathbb{H})$ is IFU-operator iff it is an isometric isomorphism of $\mathbb{H}$ on to itself. We will discuss these in detail.

2. Preliminaries

In this section, we recall some definitions and basic results which will be used throughout the paper.

Definition 2.1. ***[IFIP-Space]*** *[5] Let* $\mu : \mathcal{V}^2 \times (0, +\infty) \to [0, 1]$ *and* $\nu : \mathcal{V}^2 \times (0, +\infty) \to [0, 1]$ *be Fuzzy sets, such that* $\mu(x, y, t) + \nu(x, y, t) \leq 1, \forall x, y \in \mathcal{V} \,\&\, t > 0$. *An Intuitionistic*

Fuzzy Inner Product Space (IFIP-Space) is a triplet $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$*, where* $\mathcal{V}$ *is a real Vector Space,* $\mathcal{T}$ *is a continuous t - representable and* $\mathcal{F}_{\mu,\nu}$ *is an Intuitionistic Fuzzy set on* $\mathcal{V}^2 \times \mathbb{R}$ *satisfying the following conditions for all* $x,y,z \in \mathcal{V}$ *and* $s,r,t \in \mathbb{R}$ *:*

IFI-1: $\mathcal{F}_{\mu,\nu}(x,x,0) = 0$ *and* $\mathcal{F}_{\mu,\nu}(x,x,t) > 0$*, for every* $t > 0$.

IFI-2: $\mathcal{F}_{\mu,\nu}(x,y,t) = \mathcal{F}_{\mu,\nu}(y,x,t)$.

IFI-3: $\mathcal{F}_{\mu,\nu}(x,x,t) \neq H(t)$ *for some* $t \in \mathbb{R}$ *iff* $x \neq 0$,
where $H(t) = \begin{cases} 1, & if \quad t > 0 \\ 0, & if \quad t \leq 0 \end{cases}$

IFI-4: *For any* $\alpha \in \mathbb{R}$,

$$\mathcal{F}_{\mu,\nu}(\alpha x,y,t) = \begin{cases} \mathcal{F}_{\mu,\nu}(x,y,\frac{t}{\alpha}), & \alpha > 0 \\ H(t), & \alpha = 0 \\ \mathcal{N}_s(\mathcal{F}_{\mu,\nu}(x,y,\frac{t}{\alpha})), & \alpha < 0 \end{cases}$$

IFI-5: $\sup\{\mathcal{T}(\mathcal{F}_{\mu,\nu}(x,z,s), \mathcal{F}_{\mu,\nu}(y,z,r))\} = \mathcal{F}_{\mu,\nu}(x+y,z,t)$.

IFI-6: $\mathcal{F}_{\mu,\nu}(x,y,.) : \mathbb{R} \longrightarrow [0,1]$ *is Continuous on* $\mathbb{R} \backslash \{0\}$.

IFI-7: $\lim_{t \to 0} \mathcal{F}_{\mu,\nu}(x,y,t) = 1$.

Definition 2.2. ***[IFH-space]****[2, 5] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFIP-Space with IP:*$\langle x,y \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(x,y,t) < 1\}, \forall x,y \in \mathcal{V}$. *If* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *is complete in the norm* $\mathcal{P}_{\mu,\nu}$*, then* $\mathcal{V}$ *is an Intuitionistic Fuzzy Hilbert Space (IFH-Space).*

Theorem 2.3. ***[Riesz Theorem]****[2] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space. For any* $\tau_{\mathcal{F}_{\mu,\nu}}$ *– continuous linear functional* f*, a unique vector* $y \in \mathcal{V}$*, such that* $\forall\, x \in \mathcal{V}$*, we have* $f(x) = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(x,y,t) < 1\}$.

Definition 2.4. ***[IFA-operator]****[2] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space and let* $\mathcal{S} \in IFB(\mathcal{V})$*. Then there exists unique* $\mathcal{S}^* \in IFB(\mathcal{V}) \ni \langle \mathcal{S}x,y \rangle = \langle x, \mathcal{S}^* y \rangle, \forall x,y \in \mathcal{V}$.

Definition 2.5. ***[IFSA-operator]****[2] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle x,y \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(x,y,t) < 1\}, \forall x,y \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$*. Then* $\mathcal{S}$ *is Intuitionistic Fuzzy Self-Adjoint Operator, if* $\mathcal{S} = \mathcal{S}^*$*, where* $\mathcal{S}^*$ *is Intuitionistic Fuzzy Self-Adjoint of* $\mathcal{S}$.

Theorem 2.6. *[2] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle x,y \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(x,y,t) < 1\}, \forall x,y \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$*. Then* $\mathcal{S}$ *is Intuitionistic Fuzzy Self-Adjoint Operator.*

Definition 2.7. ***[IFN-operator]****[3] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle u,v \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(u,v,t) < 1\}, \forall u,v \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$*. Then* $\mathcal{S}$ *is Intuitionistic Fuzzy Normal Operator if it commutes with its IF-Adjoint. i.e.* $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$.

Theorem 2.8. *[3] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle u,v \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(u,v,t) < 1\}, \forall u,v \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$*. Then* $\mathcal{S}$ *is Intuitionistic Fuzzy Normal iff* $\mathcal{P}_{\mu,\nu}(\mathcal{S}^* u,t) = \mathcal{P}_{\mu,\nu}(\mathcal{S}u,t), \forall u \in \mathcal{V}$.

Theorem 2.9. *[3] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle u,v \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(u,v,t) < 1\}, \forall u,v \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$ *be an Intuitionistic Fuzzy Normal Operator. Then* $\mathcal{P}_{\mu,\nu}(\mathcal{S}^2 u,t) = \mathcal{P}^2_{\mu,\nu}(\mathcal{S}u,t)$.

Theorem 2.10. *[3] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle u,v \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(u,v,t) < 1\}, \forall u,v \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$*. Then* $\mathcal{S}$ *is Intuitionistic Fuzzy Normal iff its real and imaginary parts commute.*

Example 2.11. *[3] Let* $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ *be an IFH-Space with IP:*$\langle u,v \rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,\nu}(u,v,t) < 1\}, \forall u,v \in \mathcal{V}$ *and let* $\mathcal{S} \in IFB(\mathcal{V})$ *be an arbitrary (intuitionistic fuzzy) operator and if* γ *&* δ *are scalars such that* $|\gamma| = |\delta|$*. Then show that* $\gamma\mathcal{S} + \delta\mathcal{S}^*$ *is intuitionistic fuzzy normal.*

3. Main Results of Intuitionistic Fuzzy Unitary Operator (IFU-Operator)

In this section we introduce the definition of Intuitionistic Fuzzy Unitary operator (IFU-operator) in IFH-space and presented some elementary properties of Intuitionistic Fuzzy Unitary operator in IFH-space.

Definition 3.1. *Let* $(\mathbb{H}, \mathbb{F}_{\mu,\nu}, \mathtt{T})$ *be an IFH-space with IP:*$\langle x,y \rangle = \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x,y,t) < 1\}, \forall x,y \in \mathbb{H}$ *and let* $\mathcal{S} \in IFB(\mathbb{H})$*. Then* $\mathfrak{U}$ *is Intuitionistic Fuzzy Unitary operator if it satisfies* $\mathfrak{U}\mathfrak{U}^* = I = \mathfrak{U}^*\mathfrak{U}$.

Note 3.2.

1. *In the triplet,* $(\mathbb{H}, \mathbb{F}_{\mu,\nu}, \mathtt{T})$ *where* $\mathbb{H}$*, a real vector space,* $\mathtt{T}$ *, a continuous t-representable and* $\mathbb{F}_{\mu,\nu}$ *is an Intuitionistic Fuzzy set on* $\mathbb{H}^2 \times R$.
2. *It is obvious that, "Every Intuitionistic Fuzzy Unitary operator is Intuitionistic Fuzzy Normal".*
3. *Another definition of Intuitionistic Fuzzy Unitary operator is as follows: "An isomorphism of Intuitionistic Fuzzy Hilbert space onto itself is called Intuitionistic Fuzzy Unitary".*

Theorem 3.3. *Let* $(\mathbb{H}, \mathbb{F}_{\mu,\nu}, \mathtt{T})$ *be an IFH-space with IP:*$\langle x,y \rangle = \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x,y,t) < 1\}, \forall x,y \in \mathbb{H}$ *and let* $\mathcal{S} \in IFB(\mathbb{H})$*. If* $\mathfrak{U}$ *is Intuitionistic Fuzzy Unitary operator if and only if* $\mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t), \forall x \in \mathbb{H}$ *and* $\mathfrak{U}$ *is surjective. In that case,* $\mathcal{P}_{\mu,\nu}(\mathfrak{U}^{-1}x,t) = \mathcal{P}_{\mu,\nu}(x,t), x \in \mathbb{H}$ *&* $\mathcal{P}_{\mu,\nu}(\mathfrak{U}) = 1 = \mathcal{P}_{\mu,\nu}(\mathfrak{U}^{-1})$.

Proof. For $x \in \mathbb{H}$ we have

$$\begin{aligned}
\mathcal{P}^2_{\mu,\nu}(\mathfrak{U}x,t) - \mathcal{P}^2_{\mu,\nu}(\mathfrak{U}x,t) = \langle \mathfrak{U}x, \mathfrak{U}x \rangle - \langle x,x \rangle \\
= &\sup\{t \in R : \mathbb{F}_{\mu,\nu}(Ux,Ux,t) < 1\} \\
&- \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x,x,t) < 1\} \\
= &\sup\{t \in R : \mathbb{F}_{\mu,\nu}(U^*Ux,x,t) < 1\} \\
&- \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x,x,t) < 1\} \\
= &\langle \mathfrak{U}^*\mathfrak{U}x, x \rangle - \langle x,x \rangle \\
= &\langle (\mathfrak{U}^*\mathfrak{U} - I)x, x \rangle
\end{aligned}$$

Since $\mathfrak{U}^*\mathfrak{U} - I$ is Intuitionistic Fuzzy self-adjoint operator it follows from that $\mathfrak{U}^*\mathfrak{U} = I$ if and only if $\mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t), \forall x \in \mathbb{H}$ and if $\mathfrak{U}$ is surjective, then $\mathfrak{U}^*\mathfrak{U} = I$ and $\mathfrak{U}$ is bijective, so that

$$\mathfrak{U}\mathfrak{U}^* = (\mathfrak{U}\mathfrak{U}^*)\cdot I = (\mathfrak{U}\mathfrak{U}^*)\cdot(\mathfrak{U}\mathfrak{U}^{-1}) = \mathfrak{U}(\mathfrak{U}^*\mathfrak{U})\cdot\mathfrak{U}^{-1} = \mathfrak{U}\mathfrak{U}^{-1}$$

[Since $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary, $\mathfrak{U}\mathfrak{U}^* = I = \mathfrak{U}^*\mathfrak{U}$.]
i.e. $\mathfrak{U}\mathfrak{U}^* = I$. Therefore, $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary.

Conversely, if $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary, then $\mathfrak{U}\mathfrak{U}^* = I$ and $\mathfrak{U}^{-1} = \mathfrak{U}^*$. In that $\mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t), \forall x \in \mathbb{H}$ and $\mathfrak{U}$ is surjective, in that case it follows that $\mathcal{P}_{\mu,\nu}(\mathfrak{U}^{-1}x,t) = \mathcal{P}_{\mu,\nu}(x,t), \forall x \in \mathbb{H}$ taking the supremum over all $x \in \mathbb{H}$ with $\mathcal{P}_{\mu,\nu}(x,t) \leq 1$, we get $\mathcal{P}_{\mu,\nu}(\mathfrak{U}) = 1 = \mathcal{P}_{\mu,\nu}(\mathfrak{U}^{-1})$. □

Theorem 3.4. *If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are Intuitionistic Fuzzy Unitary operators on $(\mathbb{H}, \mathbb{F}_{\mu,\nu}, \mathtt{T})$, then $\mathfrak{U}_1\mathfrak{U}_2$ is Intuitionistic Fuzzy Unitary. $\mathfrak{U}_1 + \mathfrak{U}_2$ is Intuitionistic Fuzzy Unitary if and only if it is surjective and* $\mathrm{Re}(\mathfrak{U}_1x, \mathfrak{U}_2x) = -\frac{1}{2}, x \in \mathbb{H}$ *with* $\mathcal{P}_{\mu,\nu}(\mathfrak{U}) = 1$.

Proof. Given $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are Intuitionistic Fuzzy Unitary operators on $(\mathbb{H}, \mathbb{F}_{\mu,\nu}, \mathtt{T})$. To prove that $\mathfrak{U}_1\mathfrak{U}_2$ is Intuitionistic Fuzzy Unitary.
Let $(\mathfrak{U}_1\mathfrak{U}_2)^*\mathfrak{U}_1\mathfrak{U}_2 = \mathfrak{U}_2^*(\mathfrak{U}_1\mathfrak{U}_1^*)\mathfrak{U}_2 = \mathfrak{U}_2^*\mathfrak{U}_2$

$$(\mathfrak{U}_1\mathfrak{U}_2)^*\mathfrak{U}_1\mathfrak{U}_2 = I$$

Similarly, $\mathfrak{U}_1\mathfrak{U}_2(\mathfrak{U}_1\mathfrak{U}_2)^* = \mathfrak{U}_1(\mathfrak{U}_2\mathfrak{U}_2^*)\mathfrak{U}_1^* = \mathfrak{U}_1\mathfrak{U}_1^*$

$$\mathfrak{U}_1\mathfrak{U}_2(\mathfrak{U}_1\mathfrak{U}_2)^* = I$$

Therefore, $\mathfrak{U}_1\mathfrak{U}_2$ is Intuitionistic Fuzzy Unitary.
Assume that $\mathfrak{U}_1 + \mathfrak{U}_2$ is surjective and $\mathrm{Re}(\mathfrak{U}_1x, \mathfrak{U}_2x) = -\frac{1}{2}, x \in \mathbb{H}$ with $\mathcal{P}_{\mu,\nu}(\mathfrak{U}) = 1$.
From theorem (3.3), $\mathfrak{U}_1 + \mathfrak{U}_2$ is Intuitionistic Fuzzy Unitary.
Conversely,

$$\begin{aligned}
\mathcal{P}^2_{\mu,\nu}((\mathfrak{U}_1+\mathfrak{U}_2)x,t) &= \langle(\mathfrak{U}_1+\mathfrak{U}_2)x, (\mathfrak{U}_1+\mathfrak{U}_2)x\rangle \\
\mathcal{P}^2_{\mu,\nu}(\mathfrak{U}x,t) &= \langle\mathfrak{U}_1x, \mathfrak{U}_1x\rangle + \langle\mathfrak{U}_1x, \mathfrak{U}_2x\rangle \\
&\quad + \langle\mathfrak{U}_2x, \mathfrak{U}_1x\rangle + \langle\mathfrak{U}_2x, \mathfrak{U}_2x\rangle \\
\langle x,x\rangle &= \langle x,x\rangle + \langle\mathfrak{U}_1x, \mathfrak{U}_2x\rangle \\
&\quad + \langle\mathfrak{U}_2x, \mathfrak{U}_1x\rangle + \langle x,x\rangle \\
\langle x,x\rangle &= 2\langle x,x\rangle + 2\,\mathrm{Re}(\langle\mathfrak{U}_1x, \mathfrak{U}_2x\rangle)
\end{aligned}$$

Hence by the theorem (3.3), $\mathfrak{U}_1 + \mathfrak{U}_2$ is Intuitionistic Fuzzy Unitary if and only if it is surjective and $\langle x,x\rangle + 2\,\mathrm{Re}(\mathfrak{U}_1x + \mathfrak{U}_2x) = 0$ implies that $\mathrm{Re}(\mathfrak{U}_1x, \mathfrak{U}_2x) = -\frac{1}{2}, x \in \mathbb{H}$. □

Theorem 3.5. *If $\mathfrak{U} \in IFB(\mathbb{H})$ is Intuitionistic Fuzzy Unitary operator on $\mathbb{H}$, then the following conditions are all equivalent to one another.*

(i) $\mathfrak{U}\mathfrak{U}^* = I$.

(ii) $\langle\mathfrak{U}x, \mathfrak{U}y\rangle = \langle x,y\rangle$

(iii) $\mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t), \forall x \in \mathbb{H}$

Proof. Let $\mathfrak{U} \in IFB(\mathbb{H})$ be an Intuitionistic Fuzzy Unitary operator on $\mathbb{H}$.
(i) $\Rightarrow$ **(ii)** :
If (i) is true then

$$\begin{aligned}
\langle\mathfrak{U}x, \mathfrak{U}y\rangle &= \sup\{t \in R : \mathbb{F}_{\mu,\nu}(\mathfrak{U}x, \mathfrak{U}y, t) < 1\} \\
&= \sup\{t \in R : \mathbb{F}_{\mu,\nu}(\mathfrak{U}^*\mathfrak{U}x, y, t) < 1\} \\
&= \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x, y, t) < 1\}
\end{aligned}$$

$$\therefore \langle\mathfrak{U}x, \mathfrak{U}y\rangle = \langle x,y\rangle \text{for all} x,y \in \mathbb{H}.$$

Thus $(i) \Rightarrow (ii)$.
(ii) $\Rightarrow$ **(iii)** :
If (ii) is true then by taking $y = x$ we get

$$\begin{aligned}
\langle\mathfrak{U}x, \mathfrak{U}x\rangle &= \sup\{t \in R : \mathbb{F}_{\mu,\nu}(\mathfrak{U}x, \mathfrak{U}x, t) < 1\} \\
&= \sup\{t \in R : \mathbb{F}_{\mu,\nu}(\mathfrak{U}^*\mathfrak{U}x, x, t) < 1\} \\
&= \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x, x, t) < 1\}
\end{aligned}$$

$$\therefore \langle\mathfrak{U}x, \mathfrak{U}x\rangle = \langle x,x\rangle, \forall x,y \in \mathbb{H}.$$

$$\mathcal{P}^2_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}^2_{\mu,\nu}(x,t) \Rightarrow \mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t)$$

for all x in $\mathbb{H}$.
Thus $(ii) \Rightarrow (iii)$.
(iii) $\Rightarrow$ **(i)** :
If (iii) is true then

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t) \\
&\quad\Rightarrow \mathcal{P}^2_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}^2_{\mu,\nu}(x,t) \\
&\quad\Rightarrow \langle\mathfrak{U}x, \mathfrak{U}x\rangle = \langle x,x\rangle \\
&\quad\Rightarrow \sup\{t \in R : \mathbb{F}_{\mu,\nu}(\mathfrak{U}x, \mathfrak{U}x, t) < 1\} \\
&\qquad = \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x, x, t) < 1\} \\
&\quad\Rightarrow \sup\{t \in R : \mathbb{F}_{\mu,\nu}(\mathfrak{U}^*\mathfrak{U}x, \mathfrak{U}x, t) < 1\} \\
&\qquad = \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x, x, t) < 1\} \\
&\quad\Rightarrow \langle\mathfrak{U}^*\mathfrak{U}x, x\rangle = \langle x,x\rangle \\
&\quad\Rightarrow \langle(\mathfrak{U}^*\mathfrak{U} - I)x, x\rangle = 0 \\
&\quad\Rightarrow \mathfrak{U}^*\mathfrak{U} - I = 0 \\
&\quad\Rightarrow \mathfrak{U}^*\mathfrak{U} = I
\end{aligned}$$

Thus $(iii) \Rightarrow (i)$.
Hence the proof is complete. □

Definition 3.6. ***[Intuitionistic Fuzzy Isometric Isomorphism]*** *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two IFH-spaces. An Intuitionistic Fuzzy isometric isomorphism of $\mathbb{H}_1$ into $\mathbb{H}_2$ is a one to one linear transformation $\mathfrak{U}$ of $\mathbb{H}_1$ into $\mathbb{H}_2$ such that $\mathcal{P}_{\mu,\nu}(\mathfrak{U}x,t) = \mathcal{P}_{\mu,\nu}(x,t)$ for every $x \in \mathbb{H}_1$.*

Theorem 3.7. *Let $(\mathbb{H}, \mathbb{F}_{\mu,\nu}, \mathtt{T})$ be an IFH-space with IP:$\langle x,y\rangle = \sup\{t \in R : \mathbb{F}_{\mu,\nu}(x,y,t) < 1\}, \forall x,y \in \mathbb{H}$ and let $\mathfrak{U} \in IFB(\mathbb{H})$. If $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary operator if and only if it is an isometric isomorphism of $\mathbb{H}$ onto itself.*

Proof. Let $\mathfrak{U}$ be Intuitionistic Fuzzy Unitary operator. Then by the definition of IFU-operator, it is onto.

By theorem (3.1) it preserves Intuitionistic Fuzzy Norms, it is an isometric isomorphism of $\mathbb{H}$ on to itself.

Conversely, if $\mathfrak{U}$ is an isometric isomorphism of $\mathbb{H}$ on to itself, then $\mathfrak{U}^{-1}$ exists.

By theorem (3.1), $\mathfrak{U}^*\mathfrak{U} = I$.

Multiplying both sides by $\mathfrak{U}^{-1}$, we get

$$(\mathfrak{U}^*\mathfrak{U})\mathfrak{U}^{-1} = I \cdot \mathfrak{U}^{-1}$$
$$\Rightarrow \mathfrak{U}^* = \mathfrak{U}^{-1}$$

$\Rightarrow \mathfrak{U}\mathfrak{U}^* = I = \mathfrak{U}^*\mathfrak{U}$, which implies that $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary. □

Note 3.8. *Let $U \in IFB(\mathbb{H})$. Since $\langle \mathfrak{U}x, y\rangle = \langle x, \mathfrak{U}^*y\rangle, \forall x, y \in \mathbb{H}$, we see that $\mathfrak{U}$ is*

(a) Intuitionistic Fuzzy Normal if and only if $\langle \mathfrak{U}x, \mathfrak{U}y\rangle = \langle \mathfrak{U}^*x, \mathfrak{U}^*y\rangle$.

(b) Intuitionistic Fuzzy Unitary if and only if $\langle \mathfrak{U}x, \mathfrak{U}y\rangle = \langle x, y\rangle = \langle \mathfrak{U}^*x, \mathfrak{U}^*y\rangle$.

(c) Intuitionistic Fuzzy Self-Adjoint operator if and only if $\langle \mathfrak{U}x, y\rangle = \langle x, \mathfrak{U}y\rangle$.

We know that Intuitionistic Fuzzy Inner Product $\langle\ ,\ \rangle$ on $\mathbb{H}$ characterize the geometry of $\mathbb{H}$. Hence an operator $\mathfrak{U}$ is Intuitionistic Fuzzy Normal if $\mathfrak{U}$ and $\mathfrak{U}^*$ transforms the geometry of $\mathbb{H}$ in the same fashion. $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary if neither $\mathfrak{U}$ nor $\mathfrak{U}^*$ change the geometry of $\mathbb{H}$. For this reason, an Intuitionistic Fuzzy Unitary operator is known as an Intuitionistic Fuzzy Hilbert space isomorphism.

Theorem 3.9. *Let $\mathbb{K} = \mathbb{C}$ and $\mathfrak{U} \in IFB(\mathbb{H})$. Then there are unique Intuitionistic Fuzzy Self-Adjoint operators B and C on $\mathbb{H}$ such that $\mathfrak{U} = B + iC$. Further, $\mathfrak{U}$ is Intuitionistic Fuzzy Normal if and only if $BC = CB$, Intuitionistic Fuzzy Unitary if and only if $BC = CB$ and $B^2 + C^2 = I$ and $\mathfrak{U}$ is Intuitionistic Fuzzy Self-Adjoint operator if and only if $C = 0$.*

Proof. Let $B = \frac{\mathfrak{U}+\mathfrak{U}^*}{2}$ and $C = \frac{\mathfrak{U}-\mathfrak{U}^*}{2}$. Then B and C are Intuitionistic Fuzzy Self-Adjoint operators and $\mathfrak{U} = B + iC$. If we also have $\mathfrak{U} = B_1 + iC_1$ where B_1 and C_1 Intuitionistic Fuzzy Self-Adjoint operators, then $\mathfrak{U}^* = B_1 - iC_1$ so that

$$B_1 = \frac{\mathfrak{U}+\mathfrak{U}^*}{2} = B \text{ and } C_1 = \frac{\mathfrak{U}-\mathfrak{U}^*}{2} = C.$$

Thus, B and C are unique. Now $\mathfrak{U}$ is Intuitionistic Fuzzy Normal if and only if $(B - iC)(B + iC) = \mathfrak{U}^*\mathfrak{U} = \mathfrak{U}\mathfrak{U}^* = (B + iC)(B - iC)$. i.e. $BC = CB$.

Similarly, $\mathfrak{U}$ is Intuitionistic Fuzzy Unitary if and only if $\mathfrak{U}\mathfrak{U}^* = I = \mathfrak{U}^*\mathfrak{U}$.

$$(B + iC)(B - iC) = I = (B - iC)(B + iC)$$

$$(B^2 + C^2) + i(CB - BC) = I = (B^2 + C^2) - i(CB - BC)$$

It can be easily seen that it is equivalent to $B^2 + C^2 = I$ and $CB - BC = 0$.
Finally, $\mathfrak{U}$ is Intuitionistic Fuzzy Self-Adjoint operator if and only if $B + iC = B - iC$, i.e. $C = 0$. □

4. Conclusion

The idea of Intuitionistic fuzzy unitary operator in IFH-space is totally new and very old form of theorems play the role a prototype in our discussion of this paper. Some concepts and lemmas have been presented about Intuitionistic fuzzy unitary operator in Intuitionistic fuzzy Hilbert space. The results of this paper will be helpful for researchers to go next step in fuzzy functional analysis.

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