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On the dissipative conformable fractional singular Sturm-Liouville operator

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Abstract. In this study, a dissipative conformable fractional singular Sturm–Liouville operator is studied. For this operator, a completeness theorem is proved by Krein's theorem.

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Contents

1. Introduction and Background

In recent years, Khalil and his friends ([9]) defined conformable fractional derivative as

$$T_{\alpha}u(\zeta)=\lim_{\varepsilon\to\infty}\frac{u\left(\zeta+\varepsilon\zeta^{1-\alpha}\right)-u\left(\zeta\right)}{\varepsilon}, \tag{1.1}$$

where $0<\alpha<1$ and $u:(0,\infty)\longrightarrow\mathbb{R}:=(-\infty,\infty)$ is a function. Conformable fractional derivative aims to expand the derivative definition as known by providing the natural characteristics of classical derivative and to pain new perspectives for differential equation theory with the help of conformable differential equations obtained as using this derivative definition ([10]). Later in ([1]), Abdeljawad defined the right and left conformable fractional derivatives, the fractional chain rule and fractional integrals of higher orders.

In [2], the authors studied a conformable fractional Sturm–Liouville (CFSL) problem. In [4], Belanau et al. conctructed Weyl's theory for the conformable sequential equation with distributional potentials.

In this paper, using Krein's theorems, we prove that the system of all eigenvectors and associated vectors of dissipative CFSL operator is complete.

Now, some preliminary concepts related to conformable fractional calculus and the essentials of Krein's theorem are given.

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Definition 1.1 ([1]). *The conformable fractional entire is given by*

$$(I_\alpha u)(\zeta) = \int_0^\zeta s^{\alpha-1}u(s)ds = \int_0^\zeta u(s)d_\alpha s.$$

Let

$$L^2_\alpha(I) = \left\{ z : \left(\int_0^a |z(\zeta)|^2 \, d_\alpha \zeta \right)^{1/2} < \infty \right\},$$

where $I = [0, a)$ and $0 < a < \infty$. $L^2_\alpha(I)$ is a Hilbert space with the inner product

$$\langle u, z \rangle := \int_0^a u(\zeta)\, \overline{z(\zeta)} d_\alpha \zeta, \text{ where } u, z \in L^2_\alpha(I).$$

Theorem 1.2 (Krein [7]). *The system of root vectors of a compact dissipative operator B with nuclear imaginary component is complete in the Hilbert space H so long as at least one of the following two conditions is fulfilled:*

$$\lim_{\sigma\to\infty} \frac{n_+(\sigma, B_R)}{\sigma} = 0, \text{ or } \lim_{\sigma\to\infty} \frac{n_-(\sigma, B_R)}{\sigma} = 0,$$

where $n_+(\sigma, B_R)$ and $n_-(\sigma, B_R)$ denote the numbers of the characteristic values of the real component B_R of the operator B in the intervals $[0, \sigma]$ and $[-\sigma, 0]$, respectively.

Definition 1.3. *Let Ξ be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that*

$$|\Xi(\mu)| \le C_\varepsilon e^{\varepsilon|\mu|}, \quad \mu \in \mathbb{C} \tag{1.2}$$

then Ξ is called an entire function of the order ≤ 1 of growth and minimal type ([7]).

Theorem 1.4 ([11]). *If the entire function Ξ satisfies (1.2), then*

$$\lim_{\sigma\to\infty} \frac{n_+(\sigma, \Xi)}{\sigma} = \lim_{\sigma\to\infty} \frac{n_-(\sigma, \Xi)}{\sigma} = 0,$$

where $n_+(\sigma, \Xi)$ and $n_-(\sigma, \Xi)$ denote the numbers of the zeros of Ξ in the intervals $[0, \sigma]$ and $[-\sigma, 0]$, respectively.

2. Main Results

Consider the following singular problem

$$l[z] = -T^2_\alpha z(\zeta) + q(\zeta)z(\zeta) = \mu y, \; \zeta \in I = [0, a), \tag{2.1}$$

where q is a real-valued function on I, $q \in L^1_{\alpha,loc}(I)$, and a is a singular point.

The maximal operator is given by

$$L_{\max} z := l[z],$$

where

$$D_{\max} := \left\{ z \in L^2_\alpha(I) : z, T_\alpha z \in AC_{\alpha,loc}(I),\; l[z] \in L^2_\alpha(I) \right\}.$$

Green's formula [2] is defined as

$$\int_0^a l[z_1](\zeta)\overline{z_2(\zeta)} d_\alpha\zeta - \int_0^a z_1(\zeta)\overline{l[z_2](\zeta)} d_\alpha\zeta = [z_1, z_2](a) - [z_1, z_2](0), \tag{2.2}$$

where $z_1, z_2 \in D_{\max}$ and

$$[z_1, z_2](\zeta) = z_1(\zeta)\overline{T_\alpha z_2(\zeta)} - T_\alpha z_1(\zeta)\overline{z_2(\zeta)} = W\left(z_1, \overline{z_2}\right).$$

Set

$$D_{\min} := \left\{z \in D_{\max} : z\left(0\right) = T_\alpha z\left(0\right) = 0,\ [z, \chi](a) = 0\right\},$$

for arbitrary $\chi \in D_{\max}$.The minimal *operator* $L_{\min}$ is the restriction of $L_{\max}$ to $D_{\min}$ and $L_{\max} = L_{\min}^*$ ([4, 6, 12, 15]).

In this paper, we will assume that $L_{\min}$ has the deficiency indices $(2, 2)$.

We will denote by $\phi\left(\zeta, \mu\right), \psi\left(\zeta, \mu\right)$ two linearly independent solutions of Eq. (2.1) satisfying the following conditions

$$\begin{aligned} &\phi\left(0, \mu\right) = \cos\beta,\ T_\alpha\phi\left(0, \mu\right) = \sin\beta, \\ &\psi\left(0, \mu\right) = -\sin\beta,\ T_\alpha\psi\left(0, \mu\right) = \cos\beta, \end{aligned} \tag{2.3}$$

where $\beta \in \mathbb{R}$. $\phi\left(\zeta, \mu\right)$ and $\psi\left(\zeta, \mu\right)$ are entire functions of μ ([2]). Due to $L_{\min}$ has the deficiency indices $(2, 2)$, $\phi\left(\zeta, \mu\right), \psi\left(\zeta, \mu\right) \in L_\alpha^2(I)$.

Let $r(\zeta) = \phi\left(\zeta, 0\right)$ and $v(\zeta) = \psi(\zeta, 0)$. Then we have

$$\begin{aligned} &r\left(0\right) = \cos\beta,\ T_\alpha r\left(0\right) = \sin\beta, \\ &v\left(0\right) = -\sin\beta,\ T_\alpha v\left(0\right) = \cos\beta. \end{aligned} \tag{2.4}$$

Clearly, $r, v \in L_\alpha^2(I)$ and $r, v \in D_{\max}$.

Let

$$D(L) = \left\{z \in D_{\max} : \begin{array}{c} z\left(0\right)\cos\beta + T_\alpha z(0)\sin\beta = 0, \\ \left[z, r\right]\left(a\right) - h\left[z, v\right]\left(a\right) = 0 \end{array}\right\}, \tag{2.5}$$

where $h \in \mathbb{C}$ and $\operatorname{Im} h > 0$. Then, for all $z \in D(L)$, the operator L is defined by $Ly = l[z]$.

Theorem 2.1. *L is a dissipative operator.*

Proof. Let $z \in D(L)$. From (2.2), we find

$$\langle Lz, z\rangle - \langle z, Lz\rangle = \left[z, z\right]\left(a\right) - \left[z, z\right]\left(0\right). \tag{2.6}$$

By (2.5), we obtain

$$\left[r, z\right]\left(a\right) = -\overline{h\left[z, v\right]}\left(a\right)$$

and

$$\left[z, z\right]\left(0\right) = 0.$$

Since

$$\left[z_1, z_2\right]\left(\zeta\right) = \left[z_1, v\right]\left(\zeta\right)\left[r, z_2\right]\left(\zeta\right) - \left[z_1, r\right]\left(\zeta\right)\left[v, z_2\right]\left(\zeta\right),\ \zeta \in I, \tag{2.7}$$

where $z_1, z_2 \in D_{\max}$, we see that

$$\begin{aligned} \left[z, z\right]\left(a\right) &= \left[z, v\right]\left(a\right)\left[r, z\right]\left(a\right) - \left[z, r\right]\left(a\right)\left[v, z\right]\left(a\right) \\ &= -\overline{h}|\left[z, v\right]\left(a\right)|^2 + h|\left[z, v\right]\left(a\right)|^2 \\ &= 2i(\operatorname{Im} h)|\left[z, v\right]\left(a\right)|^2. \end{aligned}$$

Hence

$$\operatorname{Im}\langle Lz, z\rangle = (\operatorname{Im} h)|\left[z, v\right]\left(a\right)|^2 \geq 0. \tag{2.8}$$

■

Corollary 2.2. *Since the operator L is dissipative, all eigenvalues of L lie in the closed upper half-plane* $\operatorname{Im}\mu \geq 0$.

Theorem 2.3. *L has no real eigenvalue.*

Proof. Assume that μ_0 is a real eigenvalue of L and $\psi_0 = \psi_0(\zeta, \mu_0)$ is a corresponding eigenfunction. Due to

$$\operatorname{Im}(L\psi_0, \psi_0) = \operatorname{Im}(\mu_0, ||\psi_0||^2) = 0,$$

and we see that $[\psi_0, v]\,(a) = 0$. From (2.5), we obtain $[\psi_0, r]\,(a) = 0$. By (2.7), we conclude that

$$1 = W_0(\phi_0, \psi_0) = W_a(\phi_0, \psi_0) = [\phi_0, \overline{\psi_0}]\,(a)$$

$$= [\phi_0, v]\,(a)\,[r, \overline{\psi_0}]\,(a) - [\phi_0, r]\,(a)\,[\overline{v}, \psi_0]\,(a) = 0,$$

a contradiction. ■

Theorem 2.4 ([3])**.** *Every nontrivial solution z of Eq.(2.1) and $T_\alpha z$ are entire functions of μ of the order at most $\frac{1}{2}$ in the interval $[0, c]$, $c < a$.*

Let

$$\Theta_1(\mu) = [\psi(\zeta, \mu), r(\zeta)]\,(a)\,,$$
$$\Theta_2(\mu) = [\psi(\zeta, \mu), v(\zeta)]\,(a)\,,$$

where $\psi(\zeta, \mu)$ is the solution of Eq.(2.1). Clearly,

$$\sigma_p(L) = \{\mu \in \mathbb{C} : \ \Theta(\mu) = 0\}\,,$$

where

$$\Theta(\mu) = \Theta_1(\mu) - hc_2(\mu), \tag{2.9}$$

and $\sigma_p(L)$ is the point spectrum of L.

Theorem 2.5. *The functions $\Theta_1(\mu)$ and $\Theta_2(\mu)$ are entire functions of order ≤ 1 of growth and minimal type.*

Proof. Let

$$\Theta_{a_k,1}(\mu) = [\psi(\zeta, \mu), r(\zeta)]\,(a_k)\,,$$
$$\Theta_{a_k,2}(\mu) = [\psi(\zeta, \mu), v(\zeta)]\,(a_k)\,,$$

where $a_k \in I$.

By Theorem 8, $\psi(a_k, \mu)$ and $T_\alpha \psi(a_k, \mu)$ are entire functions of order $\frac{1}{2}$ for arbitrary fixed a_k. Consequently, $\Theta_{a_k,1}(\mu)$ and $\Theta_{a_k,2}(\mu)$ are entire functions of order $\frac{1}{2}$.

If we define

$$\Xi_1(\zeta, \mu) = [z, r]\,(\zeta)\,,$$
$$\Xi_2(\zeta, \mu) = [z, v]\,(\zeta)\,,$$

then we see that $\Xi_1(\zeta, \mu)$ and $\Xi_2(\zeta, \mu)$ satisfy the following system

$$T_{\alpha,\zeta}\Xi_1(\zeta, \mu) = \lambda z(\zeta, \mu) r(\zeta),\ T_{\alpha,\zeta}\Xi_2(\zeta, \mu) = \lambda z(\zeta, \mu) v(\zeta),\ \zeta \in I. \tag{2.10}$$

Using (2.10), we deduce that

$$T_{\alpha,\zeta}\Xi(\zeta, \mu) = \lambda \Omega(\zeta) \Xi(\zeta, \mu), \ \ \zeta \in I. \tag{2.11}$$

$$T_{\alpha,\zeta}\Xi(\zeta, \mu) = T_{\alpha,\zeta}\begin{bmatrix}\Xi_1(\zeta, \mu)\\ \Xi_2(\zeta, \mu)\end{bmatrix} = \begin{bmatrix}\lambda z(\zeta, \mu) r(\zeta)\\ \lambda z(\zeta, \mu) v(\zeta)\end{bmatrix}$$

$$= \mu \begin{bmatrix} [z,v]r^2 - [z,r]vr \\ [z,v]vr - [z,v]v^2 \end{bmatrix} = \mu \begin{bmatrix} \Xi_2 r^2 - \Xi_1 vr \\ \Xi_2 vr - \Xi_1 v^2 \end{bmatrix}$$

$$= \mu \begin{bmatrix} -vr & r^2 \\ -v^2 & rv \end{bmatrix} \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix},$$

where

$$\Xi(\zeta,\mu) = \begin{bmatrix} \Xi_1(\zeta,\mu) \\ \Xi_2(\zeta,\mu) \end{bmatrix},\ \Omega(\zeta) = \begin{bmatrix} -r(\zeta)v(\zeta) & r^2(\zeta) \\ -v^2(\zeta) & r(\zeta)v(\zeta) \end{bmatrix},$$

and the elements $\Omega(\zeta)$ are in $L^1_\alpha(I)$. For

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

we put $\|w\| = |w_1| + |w_2|$. The inclusion $\|\Omega(\zeta)\| \in L^1_\alpha(I)$ holds.

If $z(\zeta,\mu) = \psi(\zeta,\mu)$, then (2.11) is equivalent to the following equation

$$\Xi(\zeta,\mu) = \Xi(a_k,\mu) + \mu \int_{a_k}^{\zeta} \Omega(s)\Xi(s,\mu) d_\alpha s,\ \ \zeta \in I, \tag{2.12}$$

where

$$\Xi(a_k,\mu) = \begin{bmatrix} \Xi_1(a_k,\mu) \\ \Xi_2(a_k,\mu) \end{bmatrix} = \begin{bmatrix} [z,r]\,(a_k) \\ [z,v]\,(a_k) \end{bmatrix} = \begin{bmatrix} \Theta_{a_k,1}(\mu) \\ \Theta_{a_k,2}(\mu) \end{bmatrix},$$

$$\Xi(0,\mu) = \begin{bmatrix} \Xi_1(0,\mu) \\ \Xi_2(0,\mu) \end{bmatrix} = \begin{bmatrix} [z,r]\,(0) \\ [z,v]\,(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

$$\Xi(a,\mu) = \begin{pmatrix} \Theta_1(\mu) \\ \Theta_2(\mu) \end{pmatrix}.$$

Using Gronwall's inequality in (2.12), we find

$$\|\Xi(\zeta,\mu)\| \le \|\Xi(a_k,\mu)\| \exp\left(|\mu| \int_{a_k}^{\zeta} \|\Omega(s)\|\, d_\alpha s \right);$$

hence

$$\|\Xi(a,\mu) - \Xi(a_k,\mu)\| \le |\mu| \exp\left\{ |\mu| \int_0^a \|\Omega(s)\|\, ds \right\} \int_{a_k}^{a} \|\Omega(s)\|\, d_\alpha s, \tag{2.13}$$

$$\|\Xi(a,\mu)\| \le \exp\left(|\mu| \int_{a_k}^{a} \|\Omega(s)\|\, d_\alpha s \right) \|\Xi(a_k,\mu)\|\,. \tag{2.14}$$

(2.13) shows that $\Theta_{a_k,j}(\mu) \to \Theta_j(\mu)$ as $a_k \to a$, uniformly in μ in a compact set. Thus, $\Theta_{,j}(\mu),\ j = 1,2$, are entire functions.

By (2.14), we obtain

$$\|\Xi(a,\mu)\| \le \exp\left(|\mu| \int_0^a \|\Omega(x)\|\, d_\alpha x \right).$$

Therefore $\Theta_{,j}(\mu)$ are of not higher than the first order. By (2.14), we see that $\Theta_{,j}(\mu),\ j = 1,2$, are of minimal type. ■

From (2.2), we conclude that

$$\Theta_1(\mu) = [\psi(\zeta,\mu), r(\zeta)]\,(a) = -1 + \mu \int_0^a \psi(\zeta,\mu) r(\zeta) d_\alpha \zeta \tag{2.15}$$

$$\Theta_2(\mu) = [\psi(\zeta,\mu), v(\zeta)]\,(a) = \mu \int_0^a \psi(\zeta,\mu) v(\zeta) d_\alpha \zeta. \tag{2.16}$$

From (2.9), (2.15) and (2.16) we find that $\Theta(0) = -1$.

By Theorem 7, the inverse operator L^{-1} exists. Now we obtain L^{-1}.

Define $u(\zeta) = r(\zeta) - hv(\zeta)$. Clearly, we see that $v, u \in L^2_\alpha(I)$ and $W\,(v,u) = -1$.

Let

$$Y\Xi = \int_0^a G(\zeta,t)\Xi(x) d_\alpha t, \tag{2.17}$$

where $\Xi \in L^2_\alpha(I)$ and

$$G(\zeta,t) = \begin{cases} v(\zeta)u(t), & 0 \le \zeta \le t \\ v(t)u(\zeta) & t \le \zeta < a. \end{cases} \tag{2.18}$$

Since

$$\int_0^a \int_0^a |G(\zeta,t)|^2 \, d_\alpha t d_\alpha t < \infty,$$

we see that $Y = L^{-1}$ and the operator Y is Hilbert–Schmidt([2]). Therefore, the root lineals of L and Y coincide and the completeness of the system of all eigen- and associated functionss of L is equivalent to the completeness of those for Y.

Each eigenvector of L may have only a finite number of linear independent associated vectors due to the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite.

Theorem 2.6. *The system of all root vectors of L (also of Y) is complete in $L^2_\alpha(I)$.*

Proof. Due to $u(\zeta) = r(\zeta) - hv(\zeta)$, setting $h = h_1 + ih_2$ we get from (2.17) in view of (2.18) that $Y = Y_1 + iY_2$, where

$$Y_1\Xi = \langle G_1\,(t,\zeta)\,, \overline{\Xi\,(\zeta)}\rangle,$$

$$Y_2\Xi = \langle G_2\,(t,\zeta)\,, \overline{\Xi\,(\zeta)}\rangle$$

and

$$G_1\,(t,\zeta) = \begin{cases} v\,(t)\,[u\,(\zeta) - h_1 v\,(\zeta)], \ 0 \le \zeta \le t \le a, \\ v\,(\zeta)\,[u\,(t) - h_1 v\,(t)], \ 0 \le \zeta \le t \le a, \end{cases}$$

$$G_2\,(t,\zeta) = -h_2 v\,(t)\,v\,(\zeta)\,, \ h_2 = \operatorname{Im} h > 0.$$

Y_1 is the self-adjoint Hilbert–Schmidt operator in $L^2_\alpha(I)$, and Y_2 is the self-adjoint one-dimensional operator in $L^2_\alpha(I)$.

Let us denote by L_1 the operator in $L^2_\alpha(I)$ generated by $l[z]$ and the following conditions

$$z\,(0) \cos\alpha + T_\alpha z\,(0) \sin\alpha = 0,$$

$$[z,r]\,(a) - h_1 [z,v]\,(a) = 0,$$

where $h_1 = \operatorname{Re} h$. It is obviously that Y_1 is the inverse of the operator L_1. Let

$$\rho_p\,(L_1) = \{\mu : \mu \in \mathbb{C}, \ \Psi\,(\mu) = 0\}\,, \tag{2.19}$$

where

$$\Psi\,(\mu) := \Theta_1\,(\mu) - h_1 \Theta_2\,(\mu)\,. \tag{2.20}$$

Thus we obtain

$$|\Psi(\mu)| \leq C_\varepsilon e^{\varepsilon|\mu|},\ \forall\mu \in \mathbb{C}. \tag{2.21}$$

Let $Z = -Y$ and $Z = Z_1 + iZ_2$, where $Z_1 = -Y_1,\ Z_2 = -Y_2$. It follows from (2.19), (2.21) and Theorem 5 that

$$\lim_{\sigma\to\infty} \frac{m_+(\sigma, Z_1)}{\sigma} = 0 \text{ or } \lim_{\sigma\to\infty} \frac{m_-(\sigma, Z_1)}{\sigma} = 0,$$

where $m_+(\sigma, Z_1)$ and $m_-(\sigma, Z_1)$ denote the numbers of the characteristic values of the real component $Z_R = Z_1$ in the intervals $[0, \sigma]$ and $[-\sigma, 0]$, respectively. Thus the dissipative operator Z (also of Y) carries out all the conditions of Krein's theorem on completeness. ■

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