Graceful and felicitous labeling of Stem-Lotus graph

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Abstract. A graph obtained from a shell graph $C(2n + 3, 2n)$ where $n \geq 1$ by adding a vertex in between each pair of adjacent vertices on the cycle, adding an edge in apex and two more chords is called a Stem-Lotus graph. In this paper, I have proved that a Stem-Lotus graph is graceful and also felicitous.

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1. Introduction

We begin with simple, finite, undirected and connected graph $G = (p, q)$. An injective function $f$ from the vertices of a graph $G$ with $q$ edges to the set $\{0, 1, 2, \ldots, q\}$ is called graceful, if the edge labels induced by $|f(x) - f(y)|$ for each edge $xy$ are distinct. A graph which admits graceful labeling is called graceful graph. An injective function $f$ from the vertices of a graph $G$ with $q$ edges to the set $\{0, 1, 2, \ldots, q\}$ is called felicitous, if the edge labels induced by $(f(x) + f(y))(\text{mod } q)$ for each edge $xy$ are distinct. A graph which admits felicitous labeling is called felicitous graph.

From the excellent survey of Gallian Dec 2, 2022, one can find many families of cycle related graphs, on which Shell graph is an important family. Deb & Limaye (2002) have defined a shell graph as a cycle $C_n$ with $(n - 3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n - 3)$. In this paper, a new family of graph called a Stem-Lotus graph is introduced by adding a vertex in between each pair of adjacent vertices on the cycle of a shell graph. Here I have proved that Stem-Lotus graph is graceful and felicitous.

A graph obtained from a shell graph $C(2n + 3, 2n)$ where $n \geq 1$ by adding a vertex in between each pair of adjacent vertices on the cycle, adding an edge in apex and two more chords is called a Stem-Lotus graph. So clearly a Stem-Lotus graph must have a pollen grain ($C_4$), a stem ($P_1$), two leafs (each $P_3$) and some pairs of petals (each petal is $P_3$) as displayed in Figure 1.
In this direction, I have proved that Stem-Lotus graph with \( n \)-pairs of petals is graceful and felicitous for all \( n \geq 1 \).

2. Stem-Lotus graph with \( n \)-pairs of petals is graceful and felicitous

In this section, I have proved that Stem-Lotus graph with \( n \)-pairs of petals is graceful and felicitous for all \( n \geq 1 \).

**Theorem 2.1.** A Stem-Lotus graph with \( n \)-pairs of petals is graceful for all \( n \geq 1 \).

**Proof.** Let \( G \) be a Stem-Lotus graph as displayed in Figure 1. It is observed that, there are totally \( M = 6n + 11 \) edges and \( N = 4n + 9 \) vertices for the graph \( G \). Let \( v_0, v_1, v_2, \ldots, v_{N-1} \) be the \( N \) vertices of \( G \).

Define the vertex labeling of \( G \) as follows,

\[
\begin{align*}
    f(v_0) &= 0 \\
    f(v_{N-1}) &= \frac{3N - 5}{2} \\
    f(v_{2i}) &= \frac{N - 3 + 2i}{2}, 1 \leq i \leq \frac{N - 3}{2} \\
    f(v_{2i-1}) &= \frac{3N - 5 - 2i}{2}, 1 \leq i \leq \frac{N - 1}{2}
\end{align*}
\]

From the above vertex labeling, the set \( \{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\} \) form a monotonically increasing sequence and the sets \( \{f(v_{N-1})\} \) and \( \{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\} \) form a monotonically decreasing sequence. It is observed that,

\[
\max \left\{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\right\} < \min \left\{\{f(v_{N-1})\} \cup \{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\}\right\}.
\]

Therefore the labels of all vertices of \( G \) are distinct.

Now the edge values of \( G \) in the following manner,

- Let \( A_1 \) denote the edge \( \{v_0v_{N-1}\} \) of \( G \).
- Let \( A_2 \) denote the set of \( \frac{M+1}{3} \) edges \( \{v_0v_1, v_0v_3, v_0v_5, \ldots, v_0v_{N-4}, v_0v_{N-2}\} \) of \( G \).
- Let \( A_3 \) denote the set of \( \frac{2M-4}{3} \) edges \( \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{N-4}v_{N-3}, v_{N-3}v_{N-2}\} \) of \( G \).
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Let $A'_1, A'_2$ & $A'_3$ be the sets of observed edge values of the sets $A_1, A_2$ & $A_3$.

$$A'_1 = \{ M \}$$
$$A'_2 = \left\{ M - 1, M - 2, \ldots, \frac{2M - 1}{3} \right\}$$
$$A'_3 = \left\{ \frac{2M - 4}{3}, \frac{2M - 7}{3}, \ldots, 3, 2, 1 \right\}$$

It is observed that, $A'_1 \cup A'_2 \cup A'_3 = \{ M, M - 1, \ldots, 3, 2, 1 \}$ and the values in the sets $A'_1, A'_2$ & $A'_3$ are all distinct, hence $G$ is graceful.

**Theorem 2.2.** A Stem-Lotus graph with $n$-pairs of petals is felicitous for all $n \geq 1$.

**Proof.** Let $G$ be a Stem-Lotus graph as displayed in Figure 1. It is observed that, there are totally $M = 6n + 11$ edges and $N = 4n + 9$ vertices for the graph $G$. Let $v_0, v_1, v_2, \ldots, v_{N-1}$ be the $N$ vertices of $G$. Define the vertex labeling of $G$ as follows,

\[
\begin{align*}
  f(v_0) &= 0 \\
  f(v_{N-1}) &= N - 2 \\
  f(v_{2i}) &= \frac{N - 3 + 2i}{2}, 1 \leq i \leq \frac{N - 3}{2} \\
  f(v_{2i-1}) &= N - 2 + i, 1 \leq i \leq \frac{N - 1}{2}
\end{align*}
\]

From the above vertex labeling, the set $\{ f(v_{2i})/1 \leq i \leq \frac{N-3}{2} \}$ form a monotonically increasing sequence and the sets $\{ f(v_{N-1}) \}$ and $\{ f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2} \}$ form a monotonically decreasing sequence.

It is observed that,

$$\max \left\{ f(v_{2i})/1 \leq i \leq \frac{N - 3}{2} \right\} < \min \left\{ \{ f(v_{N-1}) \} \cup \{ f(v_{2i-1})/1 \leq i \leq \frac{N - 1}{2} \} \right\}.$$ 

Therefore the labels of all vertices of $G$ are distinct.

Now the edge values of $G$ in the following manner,

- Let $A_1$ denote the edge $\{ v_0v_{N-1} \}$ of $G$.
- Let $A_2$ denote the set of $\frac{M+1}{3}$ edges $\{ v_0v_1, v_0v_3, v_0v_5, \ldots, v_0v_{N-4}, v_0v_{N-2} \}$ of $G$.
- Let $A_3$ denote the set of $\frac{2M-4}{3}$ edges $\{ v_1v_2, v_2v_3, v_3v_4, \ldots, v_{N-4}v_{N-3}, v_{N-3}v_{N-2} \}$ of $G$.

Let $A'_1, A'_2$ & $A'_3$ be the sets of observed edge values of the sets $A_1, A_2$ & $A_3$.

$$A'_1 = \left\{ \frac{2M - 1}{3} \right\}$$
$$A'_2 = \left\{ \frac{2M + 2}{3}, \frac{2M + 5}{3}, \ldots, M - 1, 0 \right\}$$
$$A'_3 = \left\{ 1, 2, 3, \ldots, \frac{2M - 7}{3}, \frac{2M - 4}{3} \right\}$$

It is observed that, $A'_1 \cup A'_2 \cup A'_3 = \{ M - 1, M - 2, \ldots, 2, 1, 0 \}$ and the values in the sets $A'_1, A'_2$ & $A'_3$ are all distinct, hence $G$ is felicitous.

Illustrative example of the labeling given in the proof of Theorem 2.1 is displayed in Figure 2.

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Illustrative example of the labeling given in the proof of Theorem 2.2 is displayed in Figure 3.
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References


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