Some coefficient properties of a certain family of regular functions associated with lemniscate of Bernoulli and Opoola differential operator

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Abstract. In this exploration, we introduce a certain family of regular (or analytic) functions in association with the right-half of the Lemniscate of Bernoulli and the well-known Opoola differential operator. For the regular function \( f \) studied in this work, some estimates for the early coefficients, Fekete-Szegő functionals and second and third Hankel determinants are established. Another established result is the sharp upper estimate of the third Hankel determinant for the inverse function \( f^{-1} \) of \( f \).

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1. Introductory Statements

Firstly, we represent by \( A \), the family of normalized and regular functions whose form is of the Taylor’s series

\[
f(z) = z + \sum_{x=2}^{\infty} a_x z^x, \quad f(0) = f'(0) - 1 = 0
\]

and \( z \in \Sigma := \{ z \in \mathbb{C}, \text{ such that } |z| < 1 \} \). Also, represented by \( S \) is the family of functions \( f \in A \) that are also univalent in \( \Sigma \). A famous subfamily of \( S \) is the family \( ST \) of starlike functions. A function \( f \in S \) is said to be in \( ST \) if the condition \( \text{Re}(z(f'/f)) > 0 \) holds. For function class \( S \), the Koebe one-quarter theorem, see [10],

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is a famous theorem that affirms that the range of every function \( f \in \mathcal{S} \) includes the disk \( \{ w : |w| < 0.25 \} \). For this purpose, \( f \in \mathcal{S} \) has the inverse function \( f^{-1} \) where

\[
f^{-1}(f(z)) = z, \quad z \in \Sigma,
\]

\[
f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq 0.25,
\]

and some computations show that

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \tag{1.2}
\]

We represent the family of regular functions of the form

\[
\phi(z) = 1 + \sum_{x=1}^{\infty} p_x z^x, \quad z \in \Sigma \tag{1.3}
\]

by \( \mathcal{P} \) where \( \mathcal{P} \) is called the family of functions with positive real parts in \( \Sigma \). A generalization of (1.3) is the function

\[
\phi_\sigma(z) = 1 + (1 - \sigma) \sum_{x=1}^{\infty} p_x z^x, \quad z \in \Sigma, \quad 0 \leq \sigma < 1, \tag{1.4}
\]

known as the function with positive real parts of order \( \sigma \). Let \( \mathcal{P}(\sigma) \) represent the family of functions \( \phi_\sigma(z) \).

Let ”\( \prec \)” represent subordination. Then for \( f, F \in \mathcal{A} \), \( f(z) \prec F(z) \) if there exists a Schwarz function

\[
s(z) = \sum_{x=1}^{\infty} s_x z^x, \quad z \in \Sigma
\]

such that \( s(0) = 0, |s(z)| = |z| < 1 \), and \( f(z) = F(s(z)) \). Suppose \( F(z) \) is univalent in \( \Sigma \), then

\[
f(z) \prec F(z) \text{ if and only if } f(0) = F(0) \text{ and } f(\Sigma) \subset F(\Sigma).
\]

Recently, the direction of research in theory of geometric functions shows that the study of some prescribed domains \( \phi(\Sigma) \) is inexhaustible. In fact, special cases of functions \( \phi(z) \) have greatly motivated many researchers to study various kinds of natural image domains of \( \phi(\Sigma) \). Some of these domains can be found in [7, 9, 12, 13, 15, 16, 18, 21, 25–27, 29, 31] and the citations therein. Precisely, Sokol and Stankiewicz [32] reported the subfamily \( \mathcal{SL}(\ell b) \subset \mathcal{ST} \) satisfying the condition

\[
\phi(z) = z(f'/f) \prec \ell b(z) = \sqrt{1 + z}, \quad z \in \Sigma \tag{1.5}
\]

such that function \( \phi \) lies in the domain bounded by the right half of the lemniscate of Bernoulli which is geometrically represented by \( |\phi^2 - 1| < 1 \), \( \forall z \in \Sigma \). One can find some descriptive diagrams and more properties of domain \( |\phi^2 - 1| < 1 \) in [32]. The work of Lockwood [20] is a treatise of curves available for further research.

The differential operator \( D_{\tau, \mu}^{n, \beta} : \mathcal{A} \rightarrow \mathcal{A} \) was announced by Opoola [23], see also [4, 17, 27]. For \( f \in \mathcal{A} \) of the form (1.1),

\[
D_{\tau, \mu}^{0, \beta} f(z) = f(z)
\]

\[
D_{\tau, \mu}^{1, \beta} f(z) = (1 + (\beta - \mu - 1)\tau)f(z) - z \tau(\beta - \mu) + z \tau f'(z) = \mathcal{J}_\tau(f(z))
\]

\[
D_{\tau, \mu}^{2, \beta} f(z) = \mathcal{J}_\tau(D_{\tau, \mu}^{1, \beta} f(z))
\]

\[
D_{\tau, \mu}^{3, \beta} f(z) = \mathcal{J}_\tau(D_{\tau, \mu}^{2, \beta} f(z))
\]

and

\[
D_{\tau, \mu}^{n, \beta} f(z) = \mathcal{J}_\tau(D_{\tau, \mu}^{n-1, \beta} f(z))
\]
The function
\[ f \]
2. A New Family of Regular Functions

which can be simplified as

\[ D_{\tau,\mu}^{n}\beta f(z) = z + \sum_{x=2}^{\infty} (1 + (x + \beta - \mu - 1)\tau)^{n} a_{x}z^{x}, \quad z \in \Sigma \]  

(1.6)

for parameters in (2.2). It is clear that from (1.6),

1. \[ D_{\tau,\mu}^{0}\beta f(z) = D_{0,\mu}^{0}\beta f(z) = D_{0,\mu}^{0}\beta f(z) = f(z). \]

2. \[ D_{1,\mu}^{0}\beta f(z) = D_{1,\mu}^{0}\beta f(z) = D_{n}^{0}\beta f(z) \]
is the famous Sălăgean differential operator, see [3, 30].

3. \[ D_{n,\beta}^{n}\beta f(z) = D_{n,\beta}^{n}\mu f(z) = D_{n,\beta}^{n}\mu f(z) \]
is the famous Al-Oboudi differential operator, see [2].

2. A New Family of Regular Functions

The function \( f \) in \( A \) is in the family \( B_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b) \) if it satisfies the condition

\[ 1 - e^{-2i\delta\gamma^{2}z^{2}} \frac{D_{\tau,\mu}^{n+1,\beta} f(z)}{z} \prec \ell b(z) \]

(2.1)

for

\[ n \in \mathbb{N}_{0} := \mathbb{N} \cup \{0\}, \quad 0 \leq \mu \leq \beta; \quad \beta, \tau \geq 0, \quad \delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad 0 \leq \gamma \leq 1, \quad z \in \Sigma, \]

(2.2)

\( \ell b(z) \) and \( D_{\tau,\mu}^{n+1,\beta} f(z) \) are functions declared in (1.5) and (1.6), respectively. We however demonstrate that the following are special cases of \( B_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b) \). Let \( \tilde{\psi}_{0}(z) = (1 + z)/(1 - z) \) and \( \tilde{\psi}_{\sigma}(z) = (1 + (1 - 2\sigma)z)/(1 - z) \) be the extremal functions, respectively in \( P \) and \( P(\sigma) \), then

1. \( B_{\tau,\mu}^{n,\beta}(0, 0, \tilde{\psi}_{0}) = R \), the family of bounded turning functions presented in [1].

2. \( B_{\tau,\mu}^{n,\beta}(0, 0, \tilde{\psi}_{\sigma}) = R(\sigma) \), the family of bounded functions of order \( \sigma \) presented in [33] and

3. \( B_{\tau,\mu}^{n,\beta}(0, 1, \tilde{\psi}) = H \), the family of functions presented in [11].

In this investigation, a new subfamily of regular functions is defined and some estimates for early coefficients, Fekete-Szegö functional (for both real and complex parameters), and the second, and third Hankel determinants for the functions \( f \in A \) are established. We also established the upper estimate for the third Hankel determinant for the inverse function \( f^{-1} \) of \( f \in A \). We are inspired by the works in [18].

3. Lemmas

The lemmas that follow shall be needed.

Lemma 3.1 ([6]). If \( \phi(z) \in P \) and \( \alpha \in \mathbb{R} \), then

\[ p_{2} - \alpha \frac{p_{1}^{2}}{2} \leq \begin{cases} 2(1 - \alpha) \text{ when } \alpha \leq 0, \\ \frac{2}{\alpha} \text{ when } 0 \leq \alpha \leq 2, \\ 2(\alpha - 1) \text{ when } \alpha \geq 2. \end{cases} \]

Lemma 3.2 ([6]). If \( \phi(z) \in P \) and \( \beta \in \mathbb{C} \), then

\[ p_{2} - \beta \frac{p_{1}^{2}}{2} \leq 2 \max\{1, |1 - \beta|\}. \]

Lemma 3.3 ([14]). If \( \phi(z) \in P, \alpha \in \mathbb{R} \) and \( x, y \in \mathbb{N} \), then

\[ |p_{x+y} - \alpha p_{x}p_{y}| \leq \begin{cases} 2 \text{ when } 0 \leq \alpha \leq 1, \\ 2|\alpha - 1| \text{ elsewhere.} \end{cases} \]

Lemma 3.4 ([10]). If \( \phi(z) \in P, \) then \( |p_{x}| \leq 2 \) and \( x \in \mathbb{N} \).
4. Main Results

Henceforth, it is assumed that all parameters are as declared in (2.2) unless otherwise stated. Our results are therefore as follows.

4.1. Coefficient Estimates

Theorem 4.1. If $f \in B^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then

$$|a_2| \leq \frac{1}{2\phi_2}$$  (4.1)

$$|a_3| \leq \frac{13 + 8\gamma^2}{8\phi_3}$$  (4.2)

$$|a_4| \leq \frac{25 + 8\gamma^2}{16\phi_4}$$  (4.3)

$$|a_5| \leq \frac{1603 + 832\gamma^2 + 512\gamma^4}{512\phi_5}$$  (4.4)

where

$$\phi_x = (1 + (x + \beta - \mu - 1)\tau)^{n+1}.$$  (4.5)

**Proof.** Let $f \in B^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then the definition of subordination permits us to represent (2.1) as

$$(1 - e^{-2i\delta}\gamma^2 z^2)D^{n+1,\beta}_{\tau,\mu}f(z) = \ell b(s(z))$$

or

$$(1 - e^{-2i\delta}\gamma^2 z^2)(D^{n+1,\beta}_{\tau,\mu}f(z)) = z[1 + s(z)]^{1/2}.$$  (4.6)

For brevity, we use $\phi_x$ in (4.5) so that simple computation shows that (4.6) expands as

$$z + \phi_2a_2z^2 + (\phi_3a_3 - e^{-2i\delta}\gamma^2)z^3 + (\phi_4a_4 - e^{-2i\delta}\gamma^2\phi_2a_2)z^4 + (\phi_5a_5 - e^{-2i\delta}\gamma^2\phi_3a_3)z^5 + \ldots$$

$$= z + \frac{1}{4}p_1z^2 + \frac{1}{4}\left(p_2 - \frac{17}{8}p_1^2\right)z^3 + \frac{1}{4}\left(\frac{13}{32}p_1^3 - \frac{5}{4}p_1p_2 + p_3\right)z^4$$

$$+ \frac{1}{4}\left(\frac{419}{2048}p_1^4 + \frac{105}{96}p_1^2p_2 - \frac{5}{4}p_1p_3 - \frac{5}{8}p_2^2 + p_4\right)z^5 + \ldots$$  (4.7)

so that the comparison of the coefficients yields

$$a_2 = \frac{p_1}{4\phi_2}$$  (4.8)

$$a_3 = \frac{(p_2 - \frac{17}{8}p_1^2) + 4e^{-2i\delta}\gamma^2}{4\phi_3}$$  (4.9)

$$a_4 = \frac{(p_3 - \frac{5}{4}p_1p_2 + \frac{13}{32}p_1^3) + e^{-2i\delta}\gamma^2p_1}{4\phi_4}$$  (4.10)

and

$$a_5 = \frac{(p_4 - \frac{5}{4}p_1p_3 - \frac{5}{8}p_2^2 + \frac{35}{32}p_1^2p_2 + \frac{419}{2048}p_1^4) + (p_2 - \frac{17}{8}p_1^2)e^{-2i\delta}\gamma^2 + 4e^{-4i\delta}\gamma^4}{4\phi_5}.$$  (4.11)
Another commonly studied property of the coefficient problems of $f$, estimates for Fekete-Szegö functional

4.2. Estimates for Fekete-Szegö Functional

Application of triangle inequality and Lemma 3.4 in (4.8) yields our result in (4.1). Also, from (4.9),

$$|a_3| \leq \frac{|p_2 - \frac{17}{8} p_1^2| + 4|e^{-2i\delta}|\gamma^2}{4\phi_3}$$

and the application of Lemma 3.1 yields the result in (4.2). From (4.10), we have the presentation

$$|a_4| \leq \frac{|p_3 - \frac{5}{4} p_1 p_2| + \frac{13}{32} |p_3| + |e^{-2i\delta}|\gamma^2|p_1|}{4\phi_4}$$

which by the application of Lemmas 3.1 and 3.4 yields our result in (4.3). To obtain estimate for $a_5$ we have from (4.11) that

$$a_5 = \left( p_4 - \frac{5}{4} p_1 p_3 \right) - \frac{5}{8} p_2^2 \left( p_2 - \frac{7}{2} p_1^2 \right) + \frac{119}{2016} p_4^2 + (p_2 - \frac{17}{4} p_1^2) e^{-2i\delta}\gamma^2 + 4e^{-4i\delta}\gamma^4$$

and

$$|a_5| \leq \frac{|p_4 - \frac{5}{4} p_1 p_3| + \frac{5}{8} |p_2||p_2 - \frac{7}{2} p_1^2| + \frac{119}{2016} |p_4^2| + |p_2 - \frac{17}{4} p_1^2||e^{-2i\delta}|\gamma^2 + 4|e^{-4i\delta}|\gamma^4}{4\phi_5}$$

which by the application of Lemmas 3.1, 3.3 and 3.4 yields our result in (4.4).

4.2. Estimates for Fekete-Szegö Functional

Another commonly studied property of the coefficient problems of $f \in A$ is the Fekete-Szegö functional

$$FS(\epsilon, f) = |a_3 - \epsilon a_2^2|, \quad \epsilon \in \mathbb{R}$$

announced in [8]. Interested reader may see [4, 5, 17–19, 24] and the citations therein for more properties, applications, and background details.

**Theorem 4.2.** If $f \in B_{r,p}^n (\delta, \gamma, \ell b)$, then for real parameter $\epsilon$,

$$|a_3 - \epsilon a_2^2| \leq \begin{cases} \frac{1 - \alpha + 2\gamma^2}{2\phi_3} & \text{when} \quad \epsilon \leq -\frac{17\phi_2^2}{2\phi_3} \\ \frac{1 + 2\gamma^2}{2\phi_3} & \text{when} \quad -\frac{17\phi_2^2}{2\phi_3} \leq \epsilon \leq -\frac{9\phi_2^2}{2\phi_3} \\ \frac{\alpha - 1 + 2\gamma^2}{2\phi_3} & \text{when} \quad \epsilon \geq -\frac{9\phi_2^2}{2\phi_3} \end{cases}$$

(4.13)

where

$$\alpha = \frac{17\phi_3^2 + 2\epsilon \phi_3}{4\phi_2^2}.$$  

(4.14)

**Proof.** Let $\epsilon \in \mathbb{R}$. If we substitute (4.8) and (4.9) into (4.12) we will arrive at

$$|a_3 - \epsilon a_2^2| = \left| \frac{(p_2 - \frac{17}{8} p_1^2) + 4e^{-2i\delta}\gamma^2}{4\phi_3} - \frac{\epsilon p_1^2}{16\phi_2^2} \right|$$

so that

$$|a_3 - \epsilon a_2^2| \leq \frac{1}{4\phi_3} \left| p_2 - \frac{17\phi_2^2 + 2\epsilon \phi_3}{4\phi_2^2} \right| \frac{p_1^2}{2} + \left| \frac{e^{-2i\delta}\gamma^2}{\phi_3} \right|$$

or

$$|a_3 - \epsilon a_2^2| \leq \frac{1}{4\phi_3} \left| p_2 - \alpha \frac{p_1^2}{2} \right| + \frac{\gamma^2}{\phi_3}$$

where $\alpha$ is defined in (4.14). The application of Lemma 3.1 means that for $\alpha$ that satisfies conditions $\alpha \leq 0$, $0 \leq \alpha \leq 2$ and $\alpha \geq 2$, we have the results in (4.13).
Coefficient properties of a certain family of regular functions

**Theorem 4.3.** If \( f \in B_{r,n}^{\alpha,\beta}(\delta, \gamma, \ell b) \), then for complex parameter \( \xi \),

\[
|a_3 - \xi a_2^2| \leq \frac{1}{2\phi_3} \max\{1,|1-\beta|\} + \frac{\gamma^2}{\phi_3}
\]

(4.15)

where

\[
\beta = \frac{17\phi_3^2 + 2\xi\phi_3}{4\phi_3^2}
\]

(4.16)

**Proof.** Let \( \xi \in \mathbb{C} \). If we substitute (4.8) and (4.9) into (4.12) we will arrive at the inequality

\[
|a_3 - \xi a_2^2| \leq \frac{1}{4\phi_3} \left| p_2 - \left( \frac{17\phi_3^2 + 2\xi\phi_3}{4\phi_3^2} \right) \frac{p_1^2}{2} + \frac{e^{-2i\delta\pi^2}}{\phi_3} \right|
\]

or

\[
|a_3 - \xi a_2^2| \leq \frac{1}{4\phi_3} \left| p_2 - \beta \frac{p_1^2}{2} \right| + \frac{\gamma^2}{\phi_3}
\]

where \( \beta \) is defined in (4.16). The application of Lemma 3.2 produce the result in (4.15). \( \blacksquare \)

**4.3. Estimates for some Hankel Determinants**

The \( y \)-th Hankel determinant

\[
\mathcal{H}D_{y,x}(f) = \left| \begin{array}{cccc}
1 & a_{x+1} & a_{x+2} & \ldots & a_{x+y-1} \\
& a_{x+1} & a_{x+2} & \ldots & a_{x+y} \\
& & a_{x+2} & a_{x+3} & \ldots & a_{x+y+1} \\
& & & \vdots & \vdots & \vdots \\
& & & & a_{x+y-1} & a_{x+y} & \ldots & a_{x+2(y-1)}
\end{array} \right|
\]

(4.17)

\((x, y \in \mathbb{N})\) was introduced by Pommerenke [28]. (4.17) has its elements from the coefficients of \( f \) in (1.1). Observe that from (4.17), we can establish that

\[
|\mathcal{H}D_{2,1}(f)| = |a_3 - a_2^2|
\]

(4.18)

\[
|\mathcal{H}D_{2,2}(f)| = |a_2a_4 - a_3^2|
\]

(4.19)

\[
\mathcal{H}D_{3,1}(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2)
\]

(4.20)

hence,

\[
|\mathcal{H}D_{3,1}(f)| \leq |a_3||\mathcal{H}D_{2,2}(f)| + |a_4||\mathcal{G}_2(f)| + |a_5||\mathcal{H}D_{2,1}(f)|.
\]

(4.21)

where

\[
|\mathcal{G}_x(f)| = |a_xa_{x+1} - a_{x+2}|, \quad x = \{2, 3, 4, \ldots\}.
\]

(4.22)

Even though the functionals in (4.12) and (4.18) have different historical background, yet it can be observed that the functionals are related since \(|\mathcal{H}D_{2,1}(f)| = \mathcal{S}(1, f)\).

For the inverse functions \( f^{-1} \) in (1.2), Obradovic and Tuneski [22] established that

\[
|\mathcal{H}D_{3,1}(f^{-1})| = |\mathcal{H}D_{3,1}(f) - (a_3 - a_2^2)^3|
\]

(4.23)

and obtained some estimates for some subfamilies of \( \mathcal{S} \). Interested reader may see [4, 5, 17–19] and the citations therein for some properties and applications; and more background details on Hankel determinants.

**Theorem 4.4.** If \( f \in B_{r,n}^{\alpha,\beta}(\delta, \gamma, \ell b) \), then

\[
|\mathcal{H}D_{2,1}(f)| \leq \frac{1 + 2\gamma^2}{2\phi_3}
\]

(4.24)
Theorem 4.5. If \( f \in B_{r,\beta}^\phi(\delta, \gamma, \ell b) \), then
\[
|\mathcal{H}d_{2,2}(f)| \leq -4A + 8B - 2C + 8D - E + 4F + \frac{\gamma^4}{\phi_2^2}
\]  
(4.25)

where
\[
A = \frac{1}{16\phi_2^2}, \quad B = \frac{289\phi_2\phi_4 - 26\phi_3^2}{1024\phi_2^2\phi_4^2}, \quad C = \frac{1}{16\phi_3^2}, \quad \frac{17\phi_4 - 5\phi_3^2}{64\phi_2^2\phi_4^2} \quad E = \frac{1}{2\phi_3^2}\gamma^2 \quad \text{and} \quad F = \frac{17\phi_2\phi_4 + \phi_3^2}{16\phi_2^2\phi_4}\gamma^2.
\]
(4.26)

Proof. Substituting (4.8), (4.9) and (4.10) into (4.19) simplifies to
\[
\mathcal{H}d_{2,2}(f) = Ap_1p_3 - Bp_1^4 - C\ell p_2^3 + Dp_1^2p_2 - Ee^{-2i\delta}p_2 + Fe^{-2i\delta}p_1^2 - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}
\]
and for brevity we get
\[
|\mathcal{H}d_{2,2}(f)| = |Ap_1\left(p_3 - \frac{Bp_1^3}{A}\right) - C\ell p_2^2 - \frac{2Dp_1^2}{C}p_2 - Ee^{-2i\delta}p_2 - \frac{2Fp_1^2}{E}p_2 + \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}|
\]
so that
\[
|\mathcal{H}d_{2,2}(f)| \leq |Ap_1|\left|p_3 - \frac{Bp_1^3}{A}\right| + |C\ell p_2^2| - \frac{2Dp_1^2}{C}|p_2| - Ee^{-2i\delta}|p_2| - \frac{2Fp_1^2}{E}|p_2| + \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}|
\]
and the appropriate application of Lemmas 3.1, 3.3 and 3.4 yields (4.25).

Theorem 4.6. If \( f \in B_{r,\beta}^\phi(\delta, \gamma, \ell b) \), then
\[
|g_2(f)| \leq -2G + 4H + 8I + 2J
\]  
(4.27)

where
\[
G = \frac{1}{4\phi_4}, \quad H = \phi_4 + 5\phi_2\phi_3 + \frac{17\phi_4 + 13\phi_2\phi_3}{128\phi_2\phi_3\phi_4}, \quad I = \phi_4 + 13\phi_2\phi_3\phi_4, \quad \text{and} \quad J = \phi_2\phi_3 - \frac{\phi_4^2}{4\phi_2\phi_3\phi_4}\gamma^2.
\]
(4.28)

Proof. Substituting (4.8), (4.9) and (4.10) into (4.22) simplifies to
\[
g_2(f) = a_2a_3 - a_4 = -\frac{1}{4\phi_4}p_3 + \frac{\phi_4 + 5\phi_2\phi_3}{16\phi_2\phi_3\phi_4}p_1p_2 - \frac{17\phi_4 + 13\phi_2\phi_3}{128\phi_2\phi_3\phi_4}p_1^3 - \phi_2\phi_3 - \frac{\phi_4^2}{4\phi_2\phi_3\phi_4}e^{-2i\delta}\gamma^2p_1
\]
and for brevity we get
\[
g_2(f) = -Gp_3 + Hp_1p_2 - Ip_1^3 - Je^{-2i\delta}p_1
\]
for \( G, H, I \) and \( J \) in (4.28). Now some rearrangement and simplifications yield
\[
|g_2(f)| = \left|-G\left(p_3 - \frac{H}{G}p_1p_2\right) - Ip_1^3 - Je^{-2i\delta}p_1\right|
\]
so that
\[
|g_2(f)| \leq \left|-G\right|\left|p_3 - \frac{H}{G}p_1p_2\right| + |Ip_1^3| + |Je^{-2i\delta}p_1|
\]
and the appropriate application of Lemmas 3.3 and 3.4 yields (4.27).
Coefficient properties of a certain family of regular functions

Theorem 4.7. If \( f \in B_{r,\beta}^{\nu,\delta}(\delta, \gamma, \ell_b) \), then

\[
|\mathcal{H}D_{3,1}(f)| \leq \left( \frac{13 + 8\gamma^2}{8\phi_3} \right) \left[ -4A + 8B - 2C + 8D - E + 4F + \frac{\gamma^4}{\phi_2^2} \right] + \left( \frac{25 + 8\gamma^2}{16\phi_4} \right) \left[ -2G + 4H + 8I + 2J \right] + \left( \frac{1603 + 832\gamma^2 + 512\gamma^4}{512\phi_5} \right) \left[ 1 + 2\gamma^2 \right]
\]

(4.29)

where \( A, B, C, \ldots, J \) are defined in (4.26) and (4.28).

Proof. Substitute (4.2), (4.3), (4.4), (4.24), (4.25) and (4.27) into (4.21) yields (4.29).

Theorem 4.8. If \( f \in B_{r,\beta}^{\nu,\delta}(\delta, \gamma, \ell_b) \), then

\[
|\mathcal{H}D_{3,1}(f^{-1})| \leq \left( \frac{13 + 8\gamma^2}{4\phi_3} \right) \left[ -4A + 8B - 2C + 8D - E + 4F + \frac{\gamma^2}{\phi_2} \right] + \left( \frac{1 + 2\gamma^2}{2\phi_3} \right) \left[ 4L - 2K + 4N - 2M + 8P - 4R + 16Q + \frac{\gamma^4}{\phi_5} \right] + \left[ \frac{25 + 8\gamma^2}{16\phi_4} \right]^2 + \left( \frac{1}{2\phi_2} \right)^6
\]

(4.30)

where \( A, B, C, \ldots, J \) are defined in (4.26) and (4.28), and

\[
K = \frac{1}{4\phi_1}, \quad L = \frac{5}{16\phi_3}, \quad M = \frac{1}{4\phi_5} \gamma^2, \quad N = \frac{17\phi_2^2 - 6\phi_3^2}{32\phi_2^2 \phi_3 \phi_5} \gamma^2, \quad R = \frac{5}{32\phi_5}, \quad P = \frac{6\phi_2^2 + 35\phi_3^2}{128\phi_2^2 \phi_3 \phi_5}, \quad Q = \frac{816\phi_3^2 - 419\phi_4^2 \phi_3}{8192\phi_2^2 \phi_3 \phi_5} \phi_5
\]

(4.31)

Proof. Substituting (4.20) into (4.23) yields

\[
\mathcal{H}D_{3,1}(f^{-1}) = \left( a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \right) - (a_3 - a_2)^3
\]

\[
= 2a_2a_3a_4 - 2a_3^2 - a_4^2 + a_3a_5 - a_2a_5 + 3a_2a_3^2 - 3a_2a_3a_5 + a_2^6
\]

\[
= 2a_3(a_2a_4 - a_3^2) + a_5(a_3 - a_2^2) + 3a_2a_3(a_3 - a_2^2) - a_4^2 + a_6^2
\]

so that

\[
|\mathcal{H}D_{3,1}(f^{-1})| \leq 2|a_3||a_2a_4 - a_3^2|| + |a_3 - a_2^2||3a_2a_3 + a_5| + |a_4|^2 + |a_2|^6
\]

(4.32)

or

\[
|\mathcal{H}D_{3,1}(f^{-1})| \leq 2|a_3||\mathcal{H}D_{2,2}(f)| + |\mathcal{H}D_{2,1}(f)||3a_2^2a_3 + a_5| + |a_4|^2 + |a_2|^6.
\]

(4.33)

Observe that by using (4.8), (4.9) and (4.11),

\[
3a_2^2a_3 + a_5 = \frac{1}{4\phi_5} p_4 - \frac{5}{16\phi_5} p_1 p_3 + \frac{1}{4\phi_5} e^{-2i\delta} \gamma^2 p_2 - \frac{17\phi_2^2 \phi_3 - 6\phi_5}{32\phi_2^2 \phi_3 \phi_5} e^{-2i\delta} \gamma^2 p_1^2
\]

\[
- \frac{5}{32\phi_5} p_2^2 + \frac{6\phi_5 + 35\phi_2^2 \phi_3}{128\phi_2^2 \phi_3 \phi_5} p_1^2 p_3 + \frac{816\phi_3^2 - 419\phi_4^2 \phi_3}{8192\phi_2^2 \phi_3 \phi_5} p_1^4 + \frac{e^{-4i\delta} \gamma^4}{\phi_5}
\]

so that for brevity,

\[
3a_2^2a_3 + a_5 = K p_4 - L p_1 p_3 + M e^{-2i\delta} p_2 - N e^{-2i\delta} p_1^2 - R p_2^2 + P p_1^2 p_3 + Q p_1^4 + \frac{e^{-4i\delta} \gamma^4}{\phi_5}
\]

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for $K, L, M, N, R, P$ and $Q$ in (4.31). Now some rearrangement and simplifications yield

$$|3a_{2}a_{3} + a_{5}| = \left| K \left( p_{4} - \frac{L}{K} p_{1} p_{3} \right) + M e^{-2i\delta} \left( p_{2} - \frac{2N p_{1}^{2}}{M} \right) - R p_{2} \left( p_{2} - \frac{2P p_{1}^{2}}{R} \right) + Q p_{4}^{4} + \frac{e^{-4i\delta} \gamma^{4}}{\phi_{5}} \right|$$

so that

$$|3a_{2}a_{3} + a_{5}| = |K| \left| p_{4} - \frac{L}{K} p_{1} p_{3} \right| + |M e^{-2i\delta}| \left| p_{2} - \frac{2N p_{1}^{2}}{M} \right| + |R p_{2}| \left| p_{2} - \frac{2P p_{1}^{2}}{R} \right| + |Q p_{4}^{4}| + \left| \frac{e^{-4i\delta} \gamma^{4}}{\phi_{5}} \right|$$

and the appropriate application of Lemmas 3.4, 3.1 and 3.3 yields

$$|3a_{2}a_{3} + a_{5}| \leq 4L - 2K + 4N - 2M + 8P - 4R + 16Q + \frac{\gamma^{4}}{\phi_{5}}. \quad (4.34)$$

Now substituting (4.1), (4.2), (4.3), (4.24), (4.25) and (4.34) into (4.33) yields (4.30). □

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References


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