Integrity and vertex neighbor integrity of some graphs

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Abstract. The integrity $I(G)$ of a non-complete connected graph $G$ is a measure of network vulnerability and is defined by

$$I(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)|$$

where $S$ and $m(G - S)$ denote the subset of $V$ and the order of the largest component of $G - S$, respectively. The vertex neighbor integrity denoted as $\text{VNI}(G)$ is the concept of the integrity of a connected graph $G$ and is defined by

$$\text{VNI}(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)|$$

where $S$ is any vertex subversion strategy of $G$ and $m(G - S)$ is the number of vertices in the largest component of $G - S$. If a network is modelled as a graph, then the integrity number shows not only the difficulty to break down the network but also the damage that has been caused. This article includes several results on the integrity of the $k$-ary tree $H^k_n$, the diamond-necklace $N_k$, the diamond-chain $L_k$ and the thorn graph of the cycle graph and the vertex neighbor integrity of the $H^2_n$, $H^3_n$.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph. The diameter of $G$, denoted by $\text{diam}(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbor vertices of the vertex $v$ is called degree of $v$ and denoted by $\text{deg}_G(v)$. The minimum and maximum degrees of a vertex of $G$ are denoted by $\delta(G)$ and $\Delta(G)$. A vertex $v$ is said to be pendant vertex if $\text{deg}_G(v) = 1$. A vertex $u$ is called support if $u$ is adjacent to a pendant vertex [6]. The complement $\overline{G}$ of a graph $G$ is a graph whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in $G$ [6]. Let $G$ be a graph and $S \subseteq V(G)$. We denote by $< S >$ the subgraph of $G$ induced by $S$. A set $S$ is said to be an independent set of $G$, if no pair of vertices of $S$ are adjacent in $G$. The independence number of $G$, denoted by $\beta(G)$, is the cardinality of a maximum independent set of $G$. We denote by $\Omega(G)$ the set of all maximum independent sets of $G$. A vertex and an edge are said to cover each other if they are incident. A set of vertices

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which cover all the edges of a graph $G$ is called a vertex cover for $G$, while a set of edges which covers all the vertices is an edge cover. The smallest number of vertices in any vertex cover for $G$ is called its vertex covering number and is denoted by $\alpha(G)$ [6]. For any graph $G$ of order $n$, $\alpha(G) + \beta(G) = n$. For a graph $G$, we denote the minimum number of colors necessary to color $G$ by $\chi(G)$, the chromatic number of the graph $G$. The connectivity $\kappa = \kappa(G)$ of a graph $G$ is the minimum number of points whose removal results in a disconnected or trivial graph.

The vulnerability of a communication network gives us an idea of the resilience and robustness of the network after some centers in the network have failed. Vulnerability can be measured by certain parameters. In the analysis of the vulnerability of a communication network to disruption, attention is paid to the number of non-working elements and the size of the largest remaining group in it. Especially in a hostile relationship where mutual communication still takes place, it is desirable that the adversary’s network be such that the two quantities can be made small at the same time. [1]

One of the parameters used to measure the vulnerability of the graph is the integrity value. Formally, the vertex integrity (frequently called just the integrity) is

$$I(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)|$$

where $m(G - S)$ denotes the order of a largest component of $G - S$. This concept was introduced by Barefoot, Entringer and Swart [2], who discovered many of the early results on the subject. If $G$ is a graph of order $n$, then $1 \leq I(G) \leq n$ and if $H$ is any subgraph of $G$, then $I(H) \leq I(G)$. The integrity of the binomial trees was given in [8].

The concept of the vertex neighbor-integrity was introduced as a measure of graph vulnerability by M.B.Cozzens and Shu-Shih Y.Wu [4]. Let $u$ be a vertex of a graph $G = (V, E)$. Then $N(u) = \{v \in V(G), v \text{ and } u \text{ are adjacent}\}$ is the open neighborhood of $u$, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of $u$. A vertex $u$ of a graph $G$ is said to be subverted if the closed neighborhood $N[u]$ is deleted from $G$. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a vertex subversion strategy of $G$ if each of the vertices in $S$ has been subverted from $G$. Let $G - S$ be the survival subgraph when $S$ has been a vertex subversion strategy of $G$. The vertex neighbor integrity of a graph $G$, $VNI(G)$, is defined to be

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)|$$

where $S$ is any vertex subversion strategy of $G$ and $m(G - S)$ is the number of vertices in the largest component of $G - S$. The set $S$ is called the $VNI$ set of a graph $G$, which gives its neighbor integrity. The neighbor integrity for total graphs was given in [9]

In this article, we give a recursive formula on the integrity of $H_n^k$ and we calculate the integrity of the diamond-necklace $N_k$, the diamond-chain $L_k$ and the thorn graph $G^a$ of a cycle $C_n$. Then, we present a result on the vertex neighbor-integrity of $H_n^2$ and $H_n^3$.

2. Basic Results on Integrity and Vertex Neighbor Integrity

Theorem 2.1. [1] Define the comet $C_{t,r}$ to be the graph obtained by identifying one end of the path $P_t$ with the center of the star $K_{1,r}$. Then, $I(C_{t,r}) \leq I(C_{t+1,r-1}) \leq I(C_{t,r}) + 1.$

Theorem 2.2. [1] The integrity of
(a) the complete graph $K_p$ is $p$;
(b) the null graph $K_p$ is 1;
(c) the star $K_{1,n}$ is 2;
(d) the path $P_n$ is $I(P_n) = \lfloor 2\sqrt{n+1}\rfloor - 2$.
(e) the cycle $C_n$ is $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$;
(f) the comet $C_{p-r,r}$ is $I(P_p)$, if $r \leq \sqrt{p+1} - \frac{\sqrt{p}}{4}$; $\lceil 2\sqrt{p-r} \rceil - 1$, otherwise;
(g) the complete bipartite graph $K_{m,n}$ is $1 + \min(m,n)$;
(h) any complete multipartite graph of order $p$ and largest partite set of order $r$ is $p - r + 1$.

**Theorem 2.3.** [1] Let $G$ be a graph of order $p$.
(a) $I(G) = 1$ if and only if $G$ is null.
(b) $I(G) = 2$ if and only if all nontrivial components of $G$ are edges or the only nontrivial component is a star.
(c) $I(G) = p - 1$ if and only if $G$ is not complete and $G$ has girth at least 5.
(d) $I(G) = p$ if and only if $G$ is complete.

**Theorem 2.4.** [1] If in graph $G$, $v$ is a vertex for which $\deg(v) \geq I(G - v)$, then $I(G) = 1 + I(G - v)$.

Parameters that will be discussed here include the following:

- $\delta$, the minimum vertex degree;
- $\kappa$, the connectivity;
- $\alpha$, the covering number;
- $\beta$, the independence number;
- $\chi$, the chromatic number.

**Theorem 2.5.** For any graph $G$,
(a) $I(G) \leq \alpha + 1$.
(b) $I(G) \geq \delta + 1$.
(c) $I(G) \geq \chi$.
(d) $I(G) \geq (p - \kappa(G))/\beta(G) + \kappa(G)$.
(e) $I(G) = \kappa(G) + 1$ if and only if $\kappa(G) = \alpha(G)$;
(f) $I(G) = \alpha(G) + 1$ if and only if $G$ does not contain $2K_2$ as an induced subgraph;
(g) $I(G) = \delta(G) + 1$ if and only if $G \cong rK_n$ or $G \cong rK_n + F$ for some graph $F$ satisfying $\delta(F) \geq |G| - (2r - 1)n - 1$.

**Theorem 2.6.** [4] Let $P_n$ be the path on $n$ vertices. Then we have

$$VNI(P_n) = \begin{cases} \lceil 2\sqrt{n+3} \rceil - 4, & \text{if } n \geq 2 \\ 1, & \text{if } n = 1 \end{cases}$$

**Theorem 2.7.** [4] Let $C_n$ be the $n$-cycle, where $n \geq 3$. Then

$$VNI(C_n) = \begin{cases} \lceil 2\sqrt{n} \rceil - 3, & \text{if } n > 4 \\ 2, & \text{if } n = 4, \\ 1, & \text{if } n = 3. \end{cases}$$

**3. Integrity of $H_n^k$, $N_k$, $L_k$ and the Thorn Graph**

Integrity of $G$ is defined to be $I(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)|$, where $m(G - S)$ denotes the order of a largest component of $G - S$. In this section we calculated a recursive formula about the integrity of $H_n^3$ and we give a common result for the integrity of $H_n^k$. 

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Definition 3.1. [3] The complete $k$-ary tree $H^n_k$ of depth $n$ is the rooted tree in which all vertices at level $n-1$ or less have exactly $k$ children, and all vertices at level $n$ are leaves. A 2-ary tree, $H^4_2$, is illustrated in Figure 1.

Theorem 3.2. The integrity of a complete $3$-ary tree $H^n_3$ is given by

$$I(H^n_3) = \begin{cases} 
3^n + I(H^{n-2}_3), & \text{if } n \equiv 0 \text{(mod 2)} \\
3 \left\lfloor \frac{n}{2} \right\rfloor + I(H^{n-1}_3), & \text{if } n \equiv 1 \text{(mod 2)} 
\end{cases}$$

Proof. Case 1. $n \equiv 0 \text{(mod 2)}$
In this case $S$ consists of the all vertices at the level $n/2$. Hence, $m(H^n_3 - S) = \sum_{i=0}^{(n/2)-1} 3^i$, $|S| = 3^{n/2}$ and $I(H^n_3) = \sum_{i=0}^{n/2} 3^i = \frac{3^{(n/2)+1}-1}{2}$. To express this function as a recursive function we use induction. Certainly $I(H^3_4) = 3^2 + I(H^3_2) = 9 + 4 = 13$. Assume $I(H^{n-2}_3) = \frac{3^{(n-2)/2} + 1}{2}$, then we need to show that $I(H^n_3) = \frac{3^{n/2} + 1}{2}$.

$$I(H^n_3) = 3^{n/2} + I(H^{n-2}_3) \tag{3.1}$$
$$= 3^{n/2} + \frac{3^{n/2} + 1}{2} \tag{3.2}$$
$$= 3^{n/2} + \frac{3^{n/2}}{2} \tag{3.3}$$
$$= \frac{3^{n/2} + 1}{2} \tag{3.4}$$

Case 2. $n \equiv 1 \text{(mod 2)}$
In this case $S$ consists of the all vertices at the level \lfloor n/2 \rfloor. Hence, $m(H^n_3 - S) = \sum_{i=0}^{\lfloor (n/2) \rfloor} 3^i$, $|S| = 3^{\lfloor n/2 \rfloor}$ and $I(H^n_3) = \frac{3^{\lfloor (n/2) \rfloor+1}-1}{2} + 3^{\lfloor n/2 \rfloor}$. To express this function as a recursive function we use induction. Certainly $I(H^3_4) = 3 + I(H^2_3) = 3 + 4 = 7$. Assume $I(H^{n-1}_3) = \frac{3^{\lfloor (n-1)/2 \rfloor + 1}-1}{2} + 3^{\lfloor (n-1)/2 \rfloor}$, then we need to show that $I(H^n_3) = \frac{3^{\lfloor n/2 \rfloor+1}-1}{2} + 3^{\lfloor n/2 \rfloor}$.

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\[ I(H_n^3) = 3^{\lfloor n/2 \rfloor} + I(H_{n-1}^3) \]
\[ = 3^{\lfloor n/2 \rfloor} + \frac{3^{\lfloor n/2 \rfloor} - 1}{2} + 3^{\lfloor n/2 \rfloor} \]
\[ = \frac{3^{\lfloor n/2 \rfloor} - 1}{2} + 3^{\lfloor n/2 \rfloor} \]
\[ = 3^{\lfloor (n/2) + 1 \rfloor} - 1 \tag{3.8} \]

The proof is completed. \[ \blacksquare \]

**Corollary 3.3.** The integrity of a complete \( k \)-ary tree \( H_k^k \), \( k \geq 3 \) is given by

\[ I(H_k^k) = \begin{cases} \frac{k^2}{2} + I(H_{k-1}^k), & \text{if } n \equiv 0 \pmod{2} \\ \frac{k^2}{2} + I(H_{k-2}^k), & \text{if } n \equiv 1 \pmod{2} \end{cases} \]

**Definition 3.4.** [7] For \( k \geq 2 \) an integer, let \( D_1, D_2, \ldots, D_k \) be \( k \) disjoint copies of a diamond, where \( V(D_i) = \{a_i, b_i, c_i, d_i\} \) and where \( a_i b_i \) is the missing edge in \( D_i \). \( N_k \) is obtained from the disjoint union of these \( k \) diamonds by adding the edges \( \{a_i b_{i+1} \mid i = 1, 2, \ldots, k-1\} \) and adding the edge \( a_k b_1 \). We call \( N_k \) a diamond-necklace with \( k \) diamonds.

**Definition 3.5.** [7] For \( k \geq 1 \), we define a diamond-chain \( L_k \) with \( k \) diamonds as follows. Let \( L_k \) be obtained from a diamond-necklace \( N_{k+1} \) with \( k + 1 \) diamonds \( D_1, D_2, \ldots, D_{k+1} \) by removing the diamond \( D_{k+1} \) and adding two disjoint triangles \( T_1 \) and \( T_2 \) and adding a edge joining \( b_1 \) to a vertex of \( T_1 \) and adding an edge joining \( a_k \) to a vertex of \( T_2 \). A diamond-necklace, \( N_6 \), with six diamonds and a diamond-chain, \( L_2 \), with two diamonds is illustrated in Figure 2.

![Figure 2: A diamond-necklace \( N_6 \) and a diamond-chain \( L_2 \)](image_url)

**Theorem 3.6.** Let \( N_k \) be a diamond-necklace with \( k \) vertices. The integrity of \( N_k \) is \( I(N_k) = \lceil 4\sqrt{k} - 1 \rceil \).

**Proof.** For the set \( S \), we choose the vertices among \( a_i, b_i \), where \( i \in \{1, k\} \) to minimalized the value \( |S| + m(N_k - S) \). So, if we remove \( r \) vertices from \( N_k \), then we have \( r \) components. Number of vertices of a
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largest component \( m(N_k - S) \geq \frac{4k-r}{r} \).

\[
I(N_k) = \min_{S \subseteq V(N_k)} \{ |S| + m(N_k - S) \} \quad (3.9)
\]

\[
\geq \min_r \left\{ r + \frac{4k-r}{r} \right\} \quad (3.10)
\]

The function \( f(r) = r + \frac{4k-r}{r} \) takes its minimum value at \( r = 2\sqrt{k} \). We substitute the minimum value in the function \( f(r) \). As the integrity is integer valued, we round this up to get a lower bound and obtain \( I(N_k) = \lceil 4\sqrt{k} - 1 \rceil \).

\[\text{■}\]

**Theorem 3.7.** Let \( L_k \) be a diamond-chain with \( 4k + 6 \) vertices. The integrity of \( L_k \) is

\[
I(L_k) = \min_{S \subseteq V(L_k)} \{ |S| + m(L_k - S) \} \quad (3.11)
\]

\[
\geq \min_r \left\{ r + \frac{4k+6-r}{r+1} \right\} \quad (3.12)
\]

The function \( f(r) = r^2 + \frac{4k+6}{r+1} \) takes its minimum value at \( r = -1 + \sqrt{4k+7} \). We substitute the minimum value in the function \( f(r) \). As the integrity is integer valued, we round this up to get a lower bound and obtain \( I(L_k) = \lceil (1-1+\sqrt{4k+7})^2 + 4k+6 \rceil \).

\[\text{■}\]

**Definition 3.8.** [5] Let \( p_1, p_2, \ldots, p_n \) be non-negative integers and \( G \) be such a graph, \( V(G) = n \). The thorn graph of the graph \( G \), with parameters \( p_1, p_2, \ldots, p_n \), is obtained by attaching \( p_i \) new vertices of degree 1 to the vertex \( u_i \) of the graph \( G \), \( i = 1, 2, \ldots, n \). The thorn graph of the graph \( G \) will be denoted by \( G^* \) or by \( G^*(p_1, p_2, \ldots, p_n) \), if the respective parameters need to be specified. The thorn graph \( G^* \) of the cycle \( C_6 \) is illustrated in Figure 3.

![Figure 3: The thorn graph \( G^* \)](image)

**Theorem 3.9.** If \( G^* \) is a thorn graph of \( C_n \) with \( p_1 = p_2 = \ldots = p_n = p \), then \( I(G^*) = 2\lceil \sqrt{n} \sqrt{p+1} \rceil - p - 1 \).
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**Proof.** For the set $S$, we choose the vertices of order $p + 2$ in $C_n$ to minimalized the value $|S| + m(G^n - S)$. So, if we remove $r$ vertices from $C_n$, then we have $r$ components. Number of vertices of a largest component $m(C_n - S) \geq \frac{(n-r)p+n-r}{r}$.

$$I(N_k) = \min_{S \subseteq V(G^n)} \{|S| + m(G^n - S)\} \quad (3.13)$$

$$\geq \min_r \{r + \frac{(n-r)p+n-r}{r}\} \quad (3.14)$$

The function $f(r) = r + \frac{(n-r)p+n-r}{r}$ takes its minimum value at $r = \sqrt{np+n}$. Hence if we substitute the minimum value in the function $f(r)$, we have $I(C_n) = 2\sqrt{n\sqrt{p+1}} - p - 1$. Since the integrity is integer valued, we round this up to get a lower bound and obtain $I(C_n) = 2\lceil\sqrt{n\sqrt{p+1}} \rceil - p - 1$. ■

4. Vertex Neighbor Integrity of $H_n^2$ and $H_n^3$

The vertex neighbor integrity of $G$ is defined to be $VNI(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ is the largest connected component in the graph $G - S$. Here, we calculated the $VNI(H_n^2)$ and $VNI(H_n^3)$.

**Theorem 4.1.** The vertex neighbor integrity of a complete 2 - ary tree $H_n^2$ is given by

$$VNI(H_n^2) = \left\lfloor \frac{2^{n+1} - 3}{2} \right\rfloor$$

**Proof.** We have $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ vertices in $H_n^2$. If we remove $|S|$ vertices from $H_n^2$, then we have $4r + 1$ components in $H_n^2 - S$. Hence, $m(H_n^2 - S) \geq \frac{2^{n+1} - 1 - r}{4r + 1}$.

$$VNI(H_n^2) = \min_{S \subseteq V(H_n^2)} \{|S| + m(H_n^2 - S)\} \quad (4.1)$$

$$\geq \min_r \{r + \frac{2^{n+1} - 1 - r}{4r + 1}\} \quad (4.2)$$

The function $f(r) = r + \frac{2^{n+1} - 1 - r}{4r + 1} = \frac{2^{n+1} + r^2 - 1}{4r + 1}$ takes its minimum value at $r = \sqrt{2^{n+1} - 3}$. Hence if we substitute the minimum value in the function $f(r)$, we have $VNI(H_n^2) = \left\lfloor \frac{2^{n+1} - 3}{2} \right\rfloor$.

The proof is completed. ■

**Theorem 4.2.** The vertex neighbor integrity of $H_n^3$, is given by

$$VNI(H_n^3) = \begin{cases} 3^2 + VNI(H_{n-2}^3), & \text{if } n \equiv 1 \pmod{2} \\ 3^2 + VNI(H_{n-3}^3), & \text{if } n \equiv 0 \pmod{2} \end{cases}$$

**Proof.** The proof is similar to the proof of Theorem 3.2. ■

The following table gives the vertex neighbor integrity values of $H_n^2$ and $H_n^3$ for $n = 2, ..., 10$.

**Table 1:** The vertex neighbor integrity of $H_n^2$ and $H_n^3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_n^2$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>23</td>
<td>31</td>
<td>47</td>
</tr>
<tr>
<td>$H_n^3$</td>
<td>3</td>
<td>4</td>
<td>7</td>
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<td>31</td>
<td>40</td>
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<td>121</td>
<td>283</td>
</tr>
</tbody>
</table>
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