

On lacunary \mathcal{I} -invariant arithmetic convergence

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Abstract. In this study, we investigate the notion of lacunary \mathcal{I}_σ arithmetic convergence for real sequences and examine relations between this new type convergence notion and the notions of lacunary invariant arithmetic summability, lacunary strongly q -invariant arithmetic summability and lacunary σ -statistical arithmetic convergence which are defined in this study. Finally, giving the notions of lacunary \mathcal{I}_σ arithmetic statistically convergence, lacunary strongly \mathcal{I}_σ arithmetic summability, we prove the inclusion relation between them.

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1. Introduction and Background

The idea of arithmetic convergence was firstly originated by Ruckle [22]. Then, it was further investigated by many authors (for examples, see [9, 10, 34–38]).

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$, there is an integer n such that for every integer m we have $|x_m - x_{\langle m, n \rangle}| < \varepsilon$, where the symbol $\langle m, n \rangle$ denotes the greatest common divisor of two integers m and n . We denote the sequence space of all arithmetic convergent sequence by AC .

Statistical convergence of a real number sequence was firstly originated by Fast [2]. It became a notable topic in summability theory after the work of Fridy [3] and Šalát [23].

By a lacunary sequence, we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty.$$

The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$. The idea of lacunary statistical convergence was investigated by Fridy and Orhan [4] and then studied by several authors (for examples, see [5, 6, 13, 17, 27]).

In the wake of the study of ideal convergence defined by Kostyrko et al. [11], there has been comprehensive research to discover applications and summability studies of the classical theories. A lot of development have been seen in area about \mathcal{I} -convergence of sequences after the work of [1, 7, 8, 12, 16, 24, 28–30, 32].

An ideal \mathcal{I} on \mathbb{N} for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal \mathcal{I} is called admissible if \mathcal{I} contains all finite subsets of \mathbb{N} .

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A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in \mathbb{N} if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If \mathcal{I} is proper ideal of \mathbb{N} (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of \mathbb{N} , it is called the filter associated with the ideal.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in \mathbb{N} . The sequence (x_k) of elements of \mathbb{R} is said to be \mathcal{I} -convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}.$$

If (x_k) is \mathcal{I} -convergent to L , then we write $\mathcal{I} - \lim x = L$.

An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets of I , there is sequence $\{B_1, B_2, \dots\}$ of sets such that each symmetric difference $A_i \Delta B_i$ ($i = 1, 2, \dots$) is finite and $\bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional Φ on l_{∞} , the space of real bounded sequences, is said to be an invariant mean or a σ mean, if it satisfies the following conditions:

- (1) $\Phi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$;
- (2) $\Phi(e) = 1$, where $e = (1, 1, 1, \dots)$;
- (3) $\Phi(x_{\sigma(n)}) = \Phi(x_n)$ for all $(x_n) \in l_{\infty}$.

The mappings Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, Φ extends the limit functional on c , the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.

In case σ is translation mappings $\sigma(n) = n + 1$, the σ -mean is often called a Banach limit.

The space V_{σ} , the set of bounded sequences whose invariant means are equal, can be shown that

$$V_{\sigma} = \left\{ (x_k) \in l_{\infty} : \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m x_{\sigma^k(n)} = L \right\}$$

uniformly in n .

Several authors studied invariant mean and invariant convergent sequence (for examples, see [14, 15, 18–21, 25, 26, 31, 33]).

Savaş and Nuray [18] introduced the concepts of σ -statistical convergence and lacunary σ -statistical convergence and gave some inclusion relations. Nuray et al. [20] defined the concepts of σ -uniform density of subsets A of the set \mathbb{N} , \mathcal{I}_{σ} -convergence for real sequences and investigated relationships between \mathcal{I}_{σ} -convergence and invariant convergence also \mathcal{I}_{σ} -convergence and $[V_{\sigma}]_p$ -convergence. Ulusu and Nuray [33] investigated lacunary \mathcal{I} -invariant convergence and lacunary \mathcal{I} -invariant Cauchy sequence of real numbers. Recently, the concept of strong σ -convergence was generalized byavaş [25]. The concept of strongly σ -convergence was defined by Mursaleen [14].

Let θ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r = \min_n \{ |E \cap \{\sigma^m(n) : m \in I_r\}| \}$$

$$S_r = \max_n \{ |E \cap \{\sigma^m(n) : m \in I_r\}| \}.$$

If the following limits exist

$$\underline{V}_{\theta}(E) = \lim_{r \rightarrow \infty} \frac{s_r}{h_r}, \quad \overline{V}_{\theta}(E) = \lim_{r \rightarrow \infty} \frac{S_r}{h_r},$$

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then they are called a lower lacunary invariant uniform density and an upper lacunary invariant uniform density of the set E , respectively. If $\underline{V}_\theta(E) = \overline{V}_\theta(E)$, then $V_\theta(E) = \underline{V}_\theta(E) = \overline{V}_\theta(E)$ is called the lacunary invariant uniform density of E .

The class of all $E \subseteq \mathbb{N}$ with $\underline{V}_\theta(E) = 0$ will be denoted by $\mathcal{I}_{\sigma\theta}$. Note that $\mathcal{I}_{\sigma\theta}$ is an admissible ideal.

A sequence (x_m) is lacunary \mathcal{I}_σ -convergent to L , if for each $\varepsilon > 0$,

$$E(\varepsilon) := \{m \in \mathbb{N} : |x_m - L| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta},$$

i.e., $V_\theta(E(\varepsilon)) = 0$. In this case, we write $\mathcal{I}_{\sigma\theta} - \lim x_m = L$.

The arithmetic statistically convergence and lacunary arithmetic statistically convergence was examined by Yaying and Hazarika [38].

A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{m \leq t : |x_m - x_{\langle m, n \rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. We shall write $ASC - \lim x_m = x_{\langle m, n \rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m, n \rangle}$.

A sequence $x = (x_m)$ is said to be lacunary arithmetic statistically convergent if for $\varepsilon > 0$ there is an integer n such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{m \in I_r : |x_m - x_{\langle m, n \rangle}| \geq \varepsilon\}| = 0.$$

We will use $ASC_\theta - \lim x_m = x_{\langle m, n \rangle}$ to denote the sequence (x_m) is lacunary arithmetic statistically convergent to $x_{\langle m, n \rangle}$.

Kişî [9] investigated the concepts of invariant arithmetic convergence, strongly invariant arithmetic convergence, invariant arithmetic statistically convergence, lacunary invariant arithmetic statistical convergence and obtained interesting results.

In [10], arithmetic \mathcal{I} -statistically convergent sequence space and \mathcal{I} -lacunary arithmetic statistically convergent sequence space were given and established interesting results.

Kişî [10] examined \mathcal{I} -invariant arithmetic convergence, \mathcal{I}^* -invariant arithmetic convergence, q -strongly invariant arithmetic convergence of sequences.

A sequence $x = (x_p)$ is said to be invariant arithmetic convergent if for an integer n

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{p=1}^m x_{\sigma^p(s)} = x_{\langle p, n \rangle}$$

uniformly in s . In this case we write $x_p \rightarrow x_{\langle p, n \rangle} (AV_\sigma)$ and the set of all invariant arithmetic convergent sequences will be demonstrated by AV_σ .

A sequence $x = (x_p)$ is said to be strongly invariant arithmetic convergent if for an integer n

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{p=1}^m |x_{\sigma^p(s)} - x_{\langle p, n \rangle}| = 0$$

uniformly in s . In this case we write $x_p \rightarrow x_{\langle p, n \rangle} [AV_\sigma]$ to denote the sequence (x_p) is strongly invariant arithmetic convergent to $x_{\langle p, n \rangle}$ and the set of all invariant arithmetic convergent sequences will be demonstrated by $[AV_\sigma]$.

A sequence $x = (x_p)$ is said to be invariant arithmetic statistically convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{p \leq m : |x_{\sigma^p(s)} - x_{\langle p, n \rangle}| \geq \varepsilon\}| = 0$$

uniformly in s . We shall use $AS_\sigma C$ to denote the set of all invariant arithmetic statistical convergent sequences. In this case we write $AS_\sigma C - \lim x_p = x_{\langle p, \eta \rangle}$ or $x_p \rightarrow x_{\langle p, \eta \rangle} (AS_\sigma C)$.

A sequence $x = (x_p)$ is said to be lacunary invariant arithmetic statistical convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| = 0$$

uniformly in s . We shall use $AS_{\sigma\theta} C$ to denote the set of all lacunary invariant arithmetic statistical convergent sequences. In this case we write $AS_{\sigma\theta} C - \lim x_p = x_{\langle p, \eta \rangle}$.

The \mathcal{I} -invariant arithmetic convergence was defined by [10] as below:

A sequence $x = (x_p)$ is said to be \mathcal{I} -invariant arithmetic convergent if for every $\varepsilon > 0$, there is an integer η such that

$$\{p \in \mathbb{N} : |x_p - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \in \mathcal{I}_\sigma.$$

In this case we write $A\mathcal{I}_\sigma C - \lim x_p = x_{\langle p, \eta \rangle}$. We shall use $A\mathcal{I}_\sigma C$ to denote the set of all \mathcal{I} -invariant arithmetic convergent sequences.

2. Main Results

Definition 2.1. A sequence $x = (x_p)$ is said to be lacunary invariant arithmetic summable to $x_{\langle p, \eta \rangle}$ if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{p \in I_r} x_{\sigma^p(s)} = x_{\langle p, \eta \rangle},$$

uniformly in s , for an integer η .

Also, the set of lacunary strongly invariant arithmetic convergence sequences is defined as below:

$$[AV_{\sigma\theta}] = \left\{ x = (x_p) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| = 0 \right\}$$

uniformly in s . In this case, we write $x_p \rightarrow x_{\langle p, \eta \rangle} ([AV_{\sigma\theta}])$ to demonstrate the sequence (x_p) is lacunary strongly invariant arithmetic summable to $x_{\langle p, \eta \rangle}$.

Definition 2.2. A sequence $x = (x_p)$ is said to be lacunary strongly q -invariant arithmetic summable ($0 < q < \infty$) to $x_{\langle p, \eta \rangle}$ if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q = 0,$$

uniformly in s and it is indicated by $x_p \rightarrow x_{\langle p, \eta \rangle} ([AV_{\sigma\theta}]_q)$.

Definition 2.3. A sequence $x = (x_p)$ is said to be lacunary σ -statistical arithmetic convergent to $x_{\langle p, \eta \rangle}$ if for every $\varepsilon > 0$, there is an integer η such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| = 0,$$

uniformly in s .

Definition 2.4. A sequence $x = (x_p)$ is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$, if for each $\varepsilon > 0$, there is an integer η such that

$$K(\varepsilon) := \{p \in \mathbb{N} : |x_p - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta},$$

i.e., $V_\theta(K(\varepsilon)) = 0$. In this case, we write $x_p \rightarrow x_{\langle p, \eta \rangle} (A\mathcal{I}_{\sigma\theta})$ or $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p, \eta \rangle}$.

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Theorem 2.5. *Let (x_p) be bounded sequence. If (x_p) is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$, then (x_p) is lacunary invariant arithmetic summable to $x_{\langle p, \eta \rangle}$.*

Proof. Let $s \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Also, we suppose that (x_p) is bounded sequence and (x_p) is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$. Now, we estimate

$$t_\theta(s) := \left| \frac{1}{h_r} \sum_{p \in I_r} x_{\sigma^p(s)} - x_{\langle p, \eta \rangle} \right|.$$

For every $s = 1, 2, \dots$, we have

$$t_\theta(s) \leq t_\theta^1(s) + t_\theta^2(s),$$

where

$$t_\theta^1(s) := \frac{1}{h_r} \sum_{p \in I_r, |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|$$

and

$$t_\theta^2(s) := \frac{1}{h_r} \sum_{p \in I_r, |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| < \varepsilon} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|.$$

For every $s = 1, 2, \dots$, it is obvious that $t_\theta^2(s) < \varepsilon$. Since (x_p) is bounded sequence, there is a $M > 0$ such that

$$|x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \leq M, \quad (p \in I_r, s = 1, 2, \dots)$$

and so we have

$$\begin{aligned} t_\theta^1(s) &= \frac{1}{h_r} \sum_{p \in I_r, |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \\ &\leq \frac{M}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| \\ &\leq M \frac{\max_s |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}|}{h_r} = M \frac{S_r}{h_r}. \end{aligned}$$

Hence, due to our assumption, (x_p) is lacunary invariant arithmetic summable to $x_{\langle p, \eta \rangle}$. ■

In general, the converse of the Theorem 2.5 does not hold. For example, let $x = (x_p)$ be the sequence defined as follows:

$$x_p := \begin{cases} 1, & \text{if } p_{r-1} < p < p_{r-1} + [\sqrt{h_r}], \\ & \text{and } p \text{ is an even integer,} \\ 0, & \text{if } p_{r-1} < p < p_{r-1} + [\sqrt{h_r}], \\ & \text{and } p \text{ is an odd integer.} \end{cases}$$

When $\sigma(s) = s + 1$, this sequence is lacunary invariant arithmetic summable to $\frac{1}{2}$ but it is not lacunary \mathcal{I}_σ arithmetic convergent.

Now, we will give the following theorems which state relations between the notions of lacunary \mathcal{I}_σ arithmetic convergence and lacunary strongly q -invariant arithmetic summability, and we will denote that these notions are equivalent for bounded sequences.

Theorem 2.6. *If a sequence $x = (x_p)$ is lacunary strongly q -invariant arithmetic summable to $x_{\langle p, \eta \rangle}$, then it is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$.*

Proof. Let $0 < q < \infty$. Suppose that $x_p \rightarrow x_{\langle p, \eta \rangle} \left([AV_{\sigma\theta}]_q \right)$ for an integer η . Then, for every $s = 1, 2, \dots$ and $\varepsilon > 0$ we have

$$\begin{aligned} & \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q \\ & \geq \sum_{p \in I_r, |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q \\ & \geq \varepsilon^q |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| \\ & \geq \varepsilon^q \max_s |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| \end{aligned}$$

and so

$$\frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q \geq \varepsilon^q \frac{\max_s |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}|}{h_r} = \varepsilon^q \frac{S_r}{h_r}.$$

Hence, due to our assumption, $AI_{\sigma\theta} - \lim x_p = x_{\langle p, \eta \rangle}$. ■

Theorem 2.7. Let (x_p) is bounded sequence. If $x = (x_p)$ is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$, then it is lacunary strongly q -invariant arithmetic summable to $x_{\langle p, \eta \rangle}$.

Proof. Assume that $(x_p) \in l_\infty$ and $AI_{\sigma\theta} - \lim x_p = x_{\langle p, \eta \rangle}$. Let $0 < q < \infty$ and $\varepsilon > 0$. The boundedness of (x_p) implies that there exists a $M > 0$ such that $|x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \leq M$, $(p \in I_r, s = 1, 2, \dots)$. Therefore, we obtain

$$\begin{aligned} & \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q = \frac{1}{h_r} \sum_{\substack{p \in I_r \\ |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q + \frac{1}{h_r} \sum_{\substack{p \in I_r \\ |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| < \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q \\ & \leq M \frac{\max_s |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}|}{h_r} + \varepsilon^q \\ & = M \frac{S_r}{h_r} + \varepsilon^q. \end{aligned}$$

Therefore, we obtain

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}|^q = 0,$$

uniformly in s . Hence, we get $x_p \rightarrow x_{\langle p, \eta \rangle} \left([AV_{\sigma\theta}]_q \right)$. ■

Theorem 2.8. A sequence $(x_p) \in l_\infty$. Then, $x = (x_p)$ to lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$ iff it is lacunary strongly q -invariant arithmetic summable to $x_{\langle p, \eta \rangle}$.

Proof. This is an immediate consequence of Theorem 2.6 and Theorem 2.7. ■

Now, without proof, we will state a theorem that gives a relation between the notions of lacunary \mathcal{I}_σ arithmetic convergence and lacunary σ -statistical arithmetic convergence.

Theorem 2.9. A sequence $x = (x_p)$ is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$ iff this sequence is lacunary σ -statistical arithmetic convergent to $x_{\langle p, \eta \rangle}$.

Finally, introducing the notion of lacunary \mathcal{I}_σ^* arithmetic convergence, we will give the relation between this notion and the notion of lacunary \mathcal{I}_σ arithmetic convergence.

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Definition 2.10. A sequence $x = (x_p)$ is said to be lacunary \mathcal{I}_σ^* arithmetic convergent or $A\mathcal{I}_{\sigma\theta}^*$ -convergent to $x_{\langle p, \eta \rangle}$, if there exists a set $M = \{m_1 < m_2 < \dots < m_p < \dots\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ ($\mathbb{N} \setminus M = H \in \mathcal{I}_{\sigma\theta}$) and there is an integer η such that

$$\lim_{p \rightarrow \infty} x_{m_p} = x_{\langle p, \eta \rangle}.$$

In this case, we write $A\mathcal{I}_{\sigma\theta}^* - \lim x_p = x_{\langle p, \eta \rangle}$ or $x_p \rightarrow x_{\langle p, \eta \rangle} (A\mathcal{I}_{\sigma\theta}^*)$.

Theorem 2.11. If a sequence $x = (x_p)$ is lacunary \mathcal{I}_σ^* arithmetic convergent to $x_{\langle p, \eta \rangle}$, then this sequence is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$.

Proof. Let $\varepsilon > 0$. Since $A\mathcal{I}_{\sigma\theta}^* - \lim x_p = x_{\langle p, \eta \rangle}$, there exists a set $H \in \mathcal{I}_{\sigma\theta}$ such that for

$$M = \mathbb{N} \setminus H = \{m_1 < m_2 < \dots < m_p < \dots\}$$

and so there exists a $p_0 \in \mathbb{N}$ such that $|x_{m_p} - x_{\langle p, \eta \rangle}| < \varepsilon$ for every $p > p_0$. Then, for every $\varepsilon > 0$, we have

$$\begin{aligned} K(\varepsilon) &= \{p \in \mathbb{N} : |x_p - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \\ &\subset H \cup \{m_1 < m_2 < \dots < m_p < \dots\}. \end{aligned}$$

Since $\mathcal{I}_{\sigma\theta}$ is admissible ideal,

$$H \cup \{m_1 < m_2 < \dots < m_p < \dots\} \in \mathcal{I}_{\sigma\theta}$$

and so we have $K(\varepsilon) \in \mathcal{I}_{\sigma\theta}$. Hence, we get $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p, \eta \rangle}$. ■

The converse of the Theorem 2.11 holds if the ideal $\mathcal{I}_{\sigma\theta}$ has the property (AP).

Theorem 2.12. Let the ideal $\mathcal{I}_{\sigma\theta}$ be with property (AP). If a sequence $x = (x_p)$ is lacunary \mathcal{I}_σ arithmetic convergent to $x_{\langle p, \eta \rangle}$, then this sequence is lacunary \mathcal{I}_σ^* arithmetic convergent to $x_{\langle p, \eta \rangle}$.

Proof. Let the ideal $\mathcal{I}_{\sigma\theta}$ be with the property (AP) and $\varepsilon > 0$. Also, we suppose that $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p, \eta \rangle}$. Then, for every $\varepsilon > 0$ we have

$$K(\varepsilon) = \{p \in \mathbb{N} : |x_p - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta}.$$

Denote K_1, K_2, \dots, K_n as following

$$K_1 := \{p \in \mathbb{N} : |x_p - x_{\langle p, \eta \rangle}| \geq 1\}$$

and

$$K_n := \left\{ p \in \mathbb{N} : \frac{1}{n} \leq |x_p - x_{\langle p, \eta \rangle}| < \frac{1}{n-1} \right\},$$

where $n \geq 2$ ($n \in \mathbb{N}$). Note that $K_i \cap K_j = \emptyset$ ($i \neq j$) and $K_i \in \mathcal{I}_{\sigma\theta}$ (for each $i \in \mathbb{N}$). Since $\mathcal{I}_{\sigma\theta}$ has the property (AP), there exists a set sequence $\{F_n\}_{n \in \mathbb{N}}$ such that the symmetric differences $K_i \Delta F_i$ are finite (for each $i \in \mathbb{N}$) and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_{\sigma\theta}$. Now, to complete the proof, it is enough to prove that

$$\lim_{p \rightarrow \infty} x_p = x_{\langle p, \eta \rangle}, p \in M, \tag{2.1}$$

where $M = \mathbb{N} \setminus F$. Let $\gamma > 0$. Select $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \gamma$. Then, we get

$$\{p \in \mathbb{N} : |x_p - x_{\langle p, \eta \rangle}| \geq \gamma\} \subset \bigcup_{i=1}^{n+1} K_i.$$

Since the symmetric differences $K_i \Delta F_i$ ($i = 1, 2, \dots, n+1$) are finite, there exists a $p_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \left(\bigcup_{i=1}^{n+1} K_i \right) \cap \{p \in \mathbb{N} : p > p_0\} \\ &= \left(\bigcup_{i=1}^{n+1} F_i \right) \cap \{p \in \mathbb{N} : p > p_0\}. \end{aligned} \quad (2.2)$$

If $p > p_0$ and $p \notin F$, then

$$p \notin \bigcup_{i=1}^{n+1} F_i \text{ and by (2.2) } p \notin \bigcup_{i=1}^{n+1} K_i.$$

This give that

$$|x_p - x_{\langle p, \eta \rangle}| < \frac{1}{n+1} < \gamma$$

and so (2.1) holds. As a result, $AI_{\sigma\theta}^* - \lim x_p = x_{\langle p, \eta \rangle}$. ■

Definition 2.13. A sequence $x = (x_p)$ is said to be lacunary \mathcal{I} invariant arithmetic statistically convergent to $x_{\langle p, \eta \rangle}$, for each $\varepsilon > 0$ and $\delta > 0$, there is an integer η such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{p \in I_r : |x_p - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \right| \geq \delta \right\} \in \mathcal{I}_{\sigma\theta}.$$

In this case, we write $x_p \rightarrow x_{\langle p, \eta \rangle} (AI_{\sigma\theta}(S))$.

Definition 2.14. A sequence $x = (x_p)$ is said to be lacunary strongly \mathcal{I}_σ arithmetic summable to $x_{\langle p, \eta \rangle}$ if for each $\varepsilon > 0$, there is an integer η such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{p \in I_r} |x_p - x_{\langle p, \eta \rangle}| \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta}.$$

We will use $[A(\mathcal{I}_{\sigma\theta})] - \lim x_p = x_{\langle p, \eta \rangle}$ or $x_p \rightarrow x_{\langle p, \eta \rangle} ([A(\mathcal{I}_{\sigma\theta})])$ to indicate the sequence (x_m) is lacunary strongly \mathcal{I}_σ arithmetic convergent to $x_{\langle m, n \rangle}$.

Theorem 2.15. Let $\theta = \{k_r\}$ be a lacunary sequence.

- (i) If $x_p \rightarrow x_{\langle p, \eta \rangle} ([A(\mathcal{I}_{\sigma\theta})])$, then $x_p \rightarrow x_{\langle p, \eta \rangle} (AI_{\sigma\theta}(S))$.
- (ii) If $x \in l_\infty$ and $x_p \rightarrow x_{\langle p, \eta \rangle} (AI_{\sigma\theta}(S))$, then $x_p \rightarrow x_{\langle p, \eta \rangle} ([A(\mathcal{I}_{\sigma\theta})])$.
- (iii) $(AI_{\sigma\theta}(S)) \cap l_\infty = [A(\mathcal{I}_{\sigma\theta})] \cap l_\infty$.

Proof. (i) Let $\varepsilon > 0$ and $x_p \rightarrow x_{\langle p, \eta \rangle} ([A(\mathcal{I}_{\sigma\theta})])$. Then, we can write

$$\begin{aligned} \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| &\geq \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \\ &\quad |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon \\ &\geq \varepsilon \cdot \frac{1}{h_r} \left| \{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \right| \end{aligned}$$

for $s = 1, 2, \dots$. So, for any $\delta > 0$,

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\} \right| \geq \delta \right\} \\ & \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon \cdot \delta \right\} \end{aligned}$$

uniformly in s . Since $x_p \rightarrow x_{\langle p, \eta \rangle} ([A(\mathcal{I}_{\sigma\theta})])$, the set on the right-hand side belongs to $\mathcal{I}_{\sigma\theta}$ and so we obtain $x_p \rightarrow x_{\langle p, \eta \rangle} (A\mathcal{I}_{\sigma\theta}(S))$.

(ii) Suppose that $x \in l_\infty$ and $x_p \rightarrow x_{\langle p, \eta \rangle} (A\mathcal{I}_{\sigma\theta}(S))$. Then, there exists a $M > 0$ such that

$$|x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \leq M$$

for $s = 1, 2, \dots$

Given $\varepsilon > 0$, we obtain

$$\begin{aligned} \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| &= \frac{1}{h_r} \sum_{\substack{p \in I_r \\ |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| + \frac{1}{h_r} \sum_{\substack{p \in I_r \\ |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| < \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \\ &\leq \frac{M}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| + \varepsilon \end{aligned}$$

uniformly in s . Note that

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq \varepsilon\}| \geq \frac{\varepsilon}{M} \right\}.$$

It is obvious that $A(\varepsilon) \in \mathcal{I}_{\sigma\theta}$. If $r \in (A(\varepsilon))^c$ then

$$\frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| < 2\varepsilon.$$

Hence

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{p \in I_r} |x_{\sigma^p(s)} - x_{\langle p, \eta \rangle}| \geq 2\varepsilon \right\} \subset A(\varepsilon)$$

and so belongs to $\mathcal{I}_{\sigma\theta}$. This shows that $x_p \rightarrow x_{\langle p, \eta \rangle} ([A(\mathcal{I}_{\sigma\theta})])$. This completes the proof.

(iii) This is an immediate consequence of (i) ve (ii). ■

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